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Light-cone quantum mechanics of the eleven-dimensional superparticle

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Abstract: The linearized interactions of eleven-dimensional supergravity are obtained in a manifestly supersymmetric light-cone gauge formalism. These vertices are used to calculate certain one-loop processes involving external gravitini, antisymmetric three-form potentials and gravitons, thereby determining some protected terms in the effective action of M-theory compactified on a two-torus.

Keywords: M-Theory, String Duality, Supersymmetric Effective Theories.

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1. Introduction

Classical eleven-dimensional supergravity [1] is the long wavelength or low energy limit of M-theory [2]. In a number of papers [3, 4, 5, 6] it has been shown that certain one-loop quantum calculations in compactified eleven-dimensional supergravity generate terms in the effective M-theory action that arise in string theory as perturbative and non-perturbative effects. These loop calculations would have been very complicated using standard Feynman rules for the component fields, in which the many cancellations between different contributions are not at all apparent. Such cancellations would be natural in a covariant eleven-dimensional superspace formalism, but such a formalism only exists for the on-shell theory [7]. In the absence of useful eleven-dimensional covariant superspace Feynman rules, these one-loop calculations made use of a supersymmetric light-cone gauge in which sufficient supersymmetry is manifest to streamline the calculations. The purpose of this paper is to obtain the light-cone gauge Feynman rules that were used in the earlier papers and illustrate their use by evaluating some further one-loop amplitudes.

We will start from the quantum mechanical description of the massless eleven-dimensional superparticle to obtain vertex operators that describe interacting particles in linearized approximation. This will be sufficient to evaluate the one-loop Feynman diagrams by integrating over the world-lines of the circulating particles, with the vertex operator insertions representing the interactions with the external particles. This approach is modeled on the standard methods for evaluating string theory diagrams. It has also proved to be an efficient method [8] for calculating radiative corrections to Yang–Mills theories of relevance to the Standard Model [9] and S-matrix elements in $N = 8, d = 4$ supergravity [10]. In the case of the four-graviton amplitude the loop calculation reduces to the strikingly simple form of a simple kinematic factor multiplying a scalar field theory box diagram.

In section 2 the global and local symmetries of the eleven-dimensional superparticle action will be elucidated and the light-cone superspace quantum mechanics described. The light-cone superspace form of the vertex operators that describe the cubic interactions of the component fields will be introduced in section 3. As with the analogous superstring vertices these describe the emission of on-shell particles. In this case these comprise the graviton, gravitino and three-form potential with vertices that will be denoted V_h, V_ψ and $V_{C^{(3)}}$, respectively. The form of these vertex operators is uniquely determined by the requirement that they transform appropriately under the 32-component supersymmetry transformations, which form an $SO(10, 1)$ spinor. In the light-cone gauge these supersymmetries divide into sixteen linearly realized transformations and sixteen that are realized nonlinearly. The linear supersymmetry transformations are sufficiently simple to be checked completely. In order to determine the complete expressions for the vertices the only nonlinear supersymmetry transformation that needs to be checked is that of the graviton vertex. The nonlinear transformations of the other vertices are complicated and will not be considered. Certain total derivatives with respect to the world-line time parameter arise in the closure for the supersymmetry algebra which determine the time variations of the cubic interaction contributions to the light-cone supercharges. Closure of the supersymmetry algebra also generates higher-order interaction terms involving arbitrary powers of the superfields, but these will not be considered here. In section 4 dimensional reduction of the linearized eleven-dimensional theory on a circle is shown to reproduce the point particle limit of the corresponding linearized IIA superstring theory.

Section 5 will describe the use of these vertex operators to calculate certain classes of one-loop amplitudes in eleven-dimensional supergravity compactified on a circle or a two-torus. The particular processes that we will consider are ones that correspond to interactions in the effective action that are integrals over half the on-shell superspace and are very strongly constrained by supersymmetry. We will argue that the one-loop amplitudes for such 'protected' processes may be calculated completely by using the linearized (cubic) interaction vertices and do not recieve contributions from higher order contact terms. In part this will fill in details used in obtaining the one-loop results described in [3, 4] where connections to perturbative and nonperturbative D-instanton induced terms in type IIB string theory [11] were made. In addition some further amplitudes will be evaluated that lead to terms in the IIB effective action involving the third-rank and self-dual fifth-rank field strengths.

2. The massless eleven-dimensional superparticle in light-cone gauge

The configuration space of an eleven-dimensional superparticle has eleven bosonic coordinates, $X^{\hat\mu}$ ($\hat\mu = 0, 1, \ldots, 9, 11$), which form a SO(10, 1) vector and 32 fermionic Grassmann coordinates, $\Theta^{\hat A}$ ($\hat A = 1, \ldots, 32$), that form a Majorana SO(10, 1) spinor. The action for such a particle is given by

$$S_{particle} = \frac{1}{2}\int d\tau \frac{1}{e}\Pi^{\hat\mu}\Pi_{\hat\mu}\,, \tag{2.1}$$

where

$$\Pi^{\hat\mu} = \dot X^{\hat\mu} - i\overline\Theta\Gamma^{\hat\mu}\dot\Theta\,, \tag{2.2}$$

where the matrices $\Gamma^{\hat\mu}$ are 32×32-component Dirac matrices. The equations of motion that follow from this action are

$$\Pi^2 = 0\,, \qquad \dot\Pi^{\hat\mu} = 0\,, \qquad \Gamma^{\hat\mu}\Pi_{\hat\mu}\dot\Theta = 0\,. \tag{2.3}$$

The action is invariant under global Poincaré and supersymmetry transformations,

$$\delta_\alpha\Theta = \alpha\,, \qquad \delta_\alpha X^{\hat\mu} = i\bar\alpha\Gamma^{\hat\mu}\Theta\,, \tag{2.4}$$

where $\alpha^{\hat A}$ is a constant Majorana spinor parameter, as well as local reparameterizations and kappa symmetry transformations,

$$\delta_\kappa\Theta = i\Gamma^{\hat\mu}\Pi_{\hat\mu}\kappa\,, \qquad \delta_\kappa X^{\hat\mu} = i\overline\Theta\Gamma^{\hat\mu}\delta_\kappa\Theta\,, \qquad \delta_\kappa e = 4e\,\dot{\overline\Theta}\kappa\,, \tag{2.5}$$

where the parameter $\kappa^{\hat A}(\tau)$ is a Majorana spinor and a world-line scalar density.

There is no obvious way of formulating the quantum mechanics of this system covariantly but it is straightforward to quantize the system in the light-cone gauge. This is defined by using the reparameterization invariance and kappa symmetry to choose

$$X^+ = x^+ + p^+\tau\,, \qquad \Gamma^+\Theta = 0\,, \tag{2.6}$$

where the light-cone coordinates are defined by[1]

$$X^+ = \frac{1}{\sqrt{2}}(X^0 + X^9)\,, \qquad X^- = \frac{1}{\sqrt{2}}(X^0 - X^9) \tag{2.7}$$

and

$$\Gamma^+ = \frac{1}{\sqrt{2}}(\Gamma^0 + \Gamma^9)\,, \qquad \Gamma^- = \frac{1}{\sqrt{2}}(\Gamma^0 - \Gamma^9)\,. \tag{2.8}$$

The matrix Γ^+ projects onto a 16×16 subspace spanned by SO(9) Majorana spinors. In this gauge the action is expressed in terms of the transverse coordinates, $X^I(\tau)$ ($I = 1, \ldots, 8, 11$), and the 16-component SO(9) spinor, $\mathcal{S}^A$ ($A = 1, \ldots, 16$), so that

$$\mathcal{S}_{l.c.} = \int d\tau \left(\frac{1}{2} \left(\dot{X}^I \right)^2 - i\mathcal{S}\dot{\mathcal{S}} \right)\,. \tag{2.9}$$

The equations of motion, $\partial\dot{X}^I/\partial\tau = 0 = \partial\mathcal{S}^A/\partial\tau$ imply that both the momentum operator, $p^I = \dot{X}^I$ and the fermionic operator $\mathcal{S}^A$ are constant. The (anti)commutation relations that follow from the Poisson brackets are

$$[X^I, p^J] = i\delta^{IJ}\,, \qquad \{\mathcal{S}^A, \mathcal{S}^B\} = \delta^{AB}\,. \tag{2.10}$$

The 32 components of the SO(10,1) space-time supersymmetry decompose into a $\mathbf{16}_+$ of SO(9) $\times$ U(1) that is linearly realized and a $\mathbf{16}_-$ that is nonlinearly realized. The linearly realized supersymmetries are those components that satisfy $\Gamma^+\alpha = 0$ and will be associated with a spinor parameter η. They are generated by the S^A and their action on the coordinates is given by

$$\delta X^I = 0\,, \qquad \delta\mathcal{S} = \sqrt{p^+}\eta\,. \tag{2.11}$$

The nonlinearly realized supersymmetries will be associated with a second 16-component spinor parameter ϵ associated with the components that satisfy $\Gamma^-\alpha = 0$ and act as

$$\tilde{\delta}X^I = -\frac{2}{\sqrt{p^+}}\epsilon\gamma^I\mathcal{S}\,, \qquad \tilde{\delta}\mathcal{S} = \frac{i}{\sqrt{p^+}}\dot{X}^I\gamma^I\epsilon\,, \tag{2.12}$$

where γ^I is a 16×16 SO(9) gamma matrix and the $\tilde{}$ will be used to distinguish the nonlinearly realized symmetries from the linearly realized ones.

The physical states consist of 44 transverse symmetric traceless tensor states $|IJ\rangle$ (with $|IJ\rangle = |JI\rangle$ and $|II\rangle = 0$) representing the graviton, 128 gamma-tracelss spinor-vector states $|AI\rangle$ (with $\Gamma^I_{AB}|AI\rangle = 0$) representing the gravitino, and 84 antisymmetric tensor states $|LMN\rangle$ representing the three-form potential. These

[1] With this choice of light-cone directions the conventional coordinates of the IIA theory will be obtained by compactifying the X^{11} direction.

states form a representation of the linear supersymmetries generated by the action of the sixteen components of $\mathcal{S}^A$,

$$
\begin{aligned}
\mathcal{S}^A|IJ\rangle &= \Gamma^I_{AB}|BJ\rangle + \Gamma^J_{AB}|BI\rangle \\
\mathcal{S}^A|BI\rangle &= \frac{1}{4}\Gamma^J_{AB}|IJ\rangle + \frac{1}{72}\left(\Gamma^{ILMN} + 6\delta^{IL}\Gamma^{MN}\right)|LMN\rangle \\
\mathcal{S}^A|LMN\rangle &= \Gamma^{LM}_{AB}|BN\rangle + \Gamma^{MN}_{AB}|BL\rangle + \Gamma^{NL}_{AB}|BM\rangle\,.
\end{aligned}
\tag{2.13}
$$

2.1 Supersymmetry in the light-cone gauge

The lagrangean of eleven-dimensional supergravity [1] contains three fields: the graviton $h_{\hat\mu\hat\nu}$, the gravitino $\Psi^{\hat A}_{\hat\mu}$ and the three-form potential $C_{\hat\mu\hat\nu\hat\rho}$. The covariant equations of motion imply $k^{\hat\mu}k_{\hat\mu} = 0$ together with the physical state conditions,

$$
k_{\hat\mu}h^{\hat\mu}{}_{\hat\nu} = \frac{1}{2}k_{\hat\nu}h^{\hat\mu}{}_{\hat\mu}, \qquad \Gamma^{\hat\mu}\Psi_{\hat\mu} = k_{\hat\mu}\Psi^{\hat\mu} = \Gamma\cdot k\Psi_{\hat\mu} = 0\,, \qquad k^{\hat\mu}C^{(3)}_{\hat\mu\hat\nu\hat\rho} = 0\,. \tag{2.14}
$$

The light-cone gauge is reached by using the reparameterization invariance, local supersymmetry and the local symmetry associated with the three-form potential to impose the conditions

$$
h_{\hat\mu}{}^{+} = 0, \qquad \Psi^{\hat A +} = 0\,, \qquad C_{\hat\mu\hat\nu}{}^{+} = 0\,. \tag{2.15}
$$

As usual, the light-cone vertex operators will have a particularly simple form in a frame in which $k^+ = 0$, which is attainable in general when there are few enough external particles. It is also convenient to take k^- to be finite so that, with the condition $k^{\hat\mu}k_{\hat\mu} = 0$, the transverse momenta satisfy $k^I k_I = 0$, which is only possible for complexified momenta. The physical momenta can then be reached by analytic continuation. This kinematic set-up has proved useful in superstring calculations and will be general enough for our purposes. In this case the conditions (2.14) become

$$
\begin{aligned}
k_I h^I{}_{\hat\mu} = \frac{1}{2}k_{\hat\mu}h^I{}_I\,, \qquad & k_I\Psi^I = 0 = (\Gamma^J k_J - \Gamma^+ k^-)\Psi_{\hat\mu}\,, \\
\Gamma^I\Psi_I = \Gamma^+\Psi^-\,, \qquad & k^I C^{(3)}_{I\hat\mu\hat\nu} = 0\,.
\end{aligned}
\tag{2.16}
$$

The $^-$ components are determined by the constraints (2.14) for non-zero k^+ but are unrestricted when $k^+ = 0$. In writing (2.16) we have assumed that the components,

$$
h_{\hat\mu}{}^{-}\,, \qquad \Psi^{-A}, \qquad C_{IJ}{}^{-}\,, \tag{2.17}
$$

are non-infinite. Another condition that follows from (2.14) when $k^+ \neq 0$ is the tracelessness condition, $h_I{}^I = 0$, from which it follows that the physical state condition for the graviton in (2.16) is,

$$
k_I h^I{}_{\hat\mu} = 0\,. \tag{2.18}
$$

When $k^+ = 0$, the light-cone tracelessness condition does not follow from (2.14) but it can be imposed by hand.

The physical fields are classified in representations of SO(9). Thus h_{IJ} is a traceless symmetric second-rank tensor, while $h_I{}^-$ is a vector and h^{--} is a scalar. The components of the three-form potential are C_{IJK} and $C_{IJ}{}^-$. The gravitino decomposes into two SO(9) parts,

$$\Psi_{\hat{\mu}}^{\hat{A}} \equiv P^+\Psi_{\hat{\mu}}^{\hat{A}} + P^-\Psi_{\hat{\mu}}^{\hat{A}} = (\psi_{\hat{\mu}}^A, \tilde{\psi}_{\hat{\mu}}^A)\,, \tag{2.19}$$

where $P^\pm = \frac{1}{2}\Gamma^\pm\Gamma^\mp$. The transverse components $\psi_I^A, \tilde{\psi}_I^A$ are two spinor-vectors while $\psi^{-A}, \tilde{\psi}^{-A}$ are two spinors. These satisfy the constraints,

$$\begin{gathered}\gamma_{AB}^I\tilde{\psi}_I^B = 0\,, \qquad \gamma_{AB}^I\psi_I^B = \tilde{\psi}^{-A}\,, \qquad k^I\psi_I^A = 0\,, \qquad k^I\tilde{\psi}_I^A = 0\,, \\ \gamma^I k^I\tilde{\psi}_{\hat{\mu}} = 0\,, \qquad \gamma^I k^I\psi_{\hat{\mu}} = k^-\tilde{\psi}_{\hat{\mu}}\,.\end{gathered} \tag{2.20}$$

The covariant supersymmetry transformations of the component fields are given by[2]

$$\delta h_{\hat{\mu}\hat{\nu}} = \bar{\alpha}\Gamma_{(\hat{\mu}}\Psi_{\hat{\nu})}\,, \qquad \delta\Psi_{\hat{\mu}} = D_{\hat{\mu}}(\hat{\omega})\alpha + T_{\hat{\mu}}{}^{\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\chi}}\alpha\hat{F}_{\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\chi}}\,, \qquad \delta C_{\hat{\mu}\hat{\nu}\hat{\rho}}^{(3)} = \frac{3}{2}\bar{\alpha}\Gamma_{[\hat{\mu}\hat{\nu}}\Psi_{\hat{\rho}]}\,, \tag{2.21}$$

where the linearized covariant derivative is defined by

$$D_{\hat{\mu}}(\hat{\omega})\alpha = \left(k_{\hat{\mu}} + k_{[\hat{\rho}}h_{\hat{\sigma}]\hat{\mu}}\Gamma^{\hat{\rho}\hat{\sigma}}\right)\alpha \tag{2.22}$$

and the super-covariant $\hat{F}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}$ is simply the field strength $F_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} = 4k_{[\hat{\mu}}C_{\hat{\nu}\hat{\rho}\hat{\sigma}]}^{(3)}$. The tensor T is defined by

$$T^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\chi}} = \frac{1}{72}\left(\Gamma^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\chi}} - 8\Gamma^{\hat{\rho}\hat{\sigma}\hat{\chi}}\eta^{\hat{\mu}\hat{\nu}}\right), \tag{2.23}$$

where $\eta^{\hat{\mu}\hat{\nu}}$ is the Minkowski metric.

In the light-cone gauge the supersymmetry transformations (2.21) must be accompanied by compensating gauge transformations in order to preserve the gauge conditions (2.15). The compensating reparameterizations are determined by

$$\delta h_I{}^+ = \frac{1}{2}\bar{\alpha}(\Gamma^+\Psi^I + \Gamma^I\Psi^+) + k^+\xi^I + k^I\xi^+\,. \tag{2.24}$$

so that the light-cone gauge condition, $h_I^+ = 0$ is preserved by choosing

$$\xi^I = -\frac{1}{2k^+}\bar{\alpha}\Gamma^+\Psi^I\,, \qquad \xi^+ = 0\,. \tag{2.25}$$

[2]Numerical factors differ slightly from [1]. A similar normalization has been taken in [12].

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Hence the transverse components of the graviton transform in the following way,

$$\begin{aligned}\delta h_{IJ} &= \bar{\alpha}\Gamma_{(I}\Psi_{J)} - \frac{1}{k^+}\bar{\alpha}\Gamma^+ k_{(I}\Psi_{J)} \\ &= \eta\gamma_{(I}\tilde{\psi}_{J)} + \epsilon\gamma_{(I}\psi_{J)} - \sqrt{2}\epsilon\frac{k_{(I}\tilde{\psi}_{J)}}{k^+}\,,\end{aligned} \tag{2.26}$$

where $\psi_I, \tilde{\psi}_I, \eta$ and ϵ are SO(9) Majorana spinors defined by

$$\begin{aligned}\psi_I &= \frac{1}{2}\Gamma^+\Gamma^-\Psi_I\,, \qquad & \tilde{\psi}_I &= \frac{1}{2}\Gamma^-\Gamma^+\Psi_I\,, \\ \eta &= \frac{1}{2}\Gamma^+\Gamma^-\alpha\,, \qquad & \epsilon &= \frac{1}{2}\Gamma^-\Gamma^+\alpha\,.\end{aligned}$$

The reparametrization transformations on Ψ_I and $C^{(3)}_{LMN}$ vanish in the free-field limit and supersymmetry transformations for these fields in (2.21) are to be compensated by gauge transformations,

$$\delta\Psi_{\hat{\mu}} = k_{\hat{\mu}}\beta\,, \qquad \delta C^{(3)}_{\hat{\mu}\hat{\nu}\hat{\rho}} = 3k_{[\hat{\mu}}\upsilon_{\hat{\nu}\hat{\rho}]}\,, \tag{2.27}$$

where the gauge parameters β and $\upsilon^{\hat{\mu}\hat{\nu}}$ are an SO(10, 1) Majorana spinor and a two-form, respectively. Their compensating gauge transformations are

$$\beta = -\alpha - \frac{1}{k^+}T^{+\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}\alpha F_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}, \qquad \upsilon_{IJ} = -\frac{1}{k^+}\bar{\alpha}\Gamma^+{}_{[I}\Psi_{J]}\,, \qquad \upsilon^+_I = 0\,. \tag{2.28}$$

Hence the transverse components of the gravitino transform in the following way

$$\delta\Psi_I = D_I\alpha + \frac{1}{72}(\Gamma_I{}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}F_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} - 8\Gamma^{\hat{\mu}\hat{\nu}\hat{\rho}}F_{I\hat{\mu}\hat{\nu}\hat{\rho}})\alpha - k_I\alpha - \frac{k_I}{k^+}T^{+\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}\alpha F_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}\,, \tag{2.29}$$

and

$$\delta C^{(3)}_{LMN} = \frac{3}{2}\bar{\alpha}\Gamma_{[LM}\Psi_{N]} - \frac{6}{k^+}\bar{\alpha}k_{[L}\Gamma^+_M\Psi_{N]}\,. \tag{2.30}$$

In terms of SO(9) spinors,

$$\begin{aligned}\delta\psi_I &= k_{[L}h_{M]I}\gamma^{LM}\eta + \frac{1}{72}\left(\gamma_I{}^{JLMN}F_{JLMN} + 24\gamma^{LMN}k_L C^{(3)}_{IMN} - 4\gamma^{LMN}k_I C^{(3)}_{LMN}\right)\eta - \\ &\quad - k_{[+}h_{J]I}\gamma^J\epsilon - \frac{\sqrt{2}}{72}\frac{k_I}{k^+}\gamma^{JLMN}F_{JLMN}\,\epsilon + \cdots\,, \\ \delta\tilde{\psi}_I &= -\frac{\sqrt{2}}{2}k^+h_{IJ}\gamma^J\eta + \frac{\sqrt{2}}{18}k^+(\gamma^{ILMN} - 6\gamma^{MN}\delta^{IL})C^{(3)}_{LMN}\,\eta + k_{[L}h_{M]I}\gamma^{LM}\epsilon + \\ &\quad + \frac{1}{72}\left(\gamma_I{}^{JLMN}F_{JLMN} + 24\gamma^{LMN}k_L C^{(3)}_{IMN} + 4\gamma^{LMN}k_I C^{(3)}_{LMN}\right)\,\epsilon + \cdots\,,\end{aligned} \tag{2.31}$$

where $\cdots$ indicate terms with longitudinal polarizations and

$$\delta C^{(3)}_{LMN} = \frac{3}{2}\eta\gamma_{[LM}\tilde{\psi}_{N]} + \frac{3}{2}\epsilon\gamma_{[LM}\psi_{N]} - \frac{6\sqrt{2}}{k^+}\epsilon k_{[L}\gamma_M\tilde{\psi}_{N]}\,. \tag{2.32}$$

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3. Supersymmetry and the vertex operators

Before showing in some detail how the vertex operators are determined by the requirement that they form a representation of the supersymmetry algebra we will summarize the results. The vertex operator describing the emission of an on-shell physical field, Φ, will be written in the form

$$V_\Phi = U_\Phi e^{-ik\cdot X} = U_\Phi e^{-ik_I X^I(\tau)} e^{ik^- p^+ \tau}\,, \tag{3.1}$$

where the prefactor U_Φ depends on the species of field.

The following notation will also be introduced,

$$\mathcal{R}^{LMN} = \frac{1}{12}\mathcal{S}\gamma^{LMN}\mathcal{S}\,, \qquad \mathcal{R}^{IL} = \frac{1}{4}\mathcal{S}\gamma^{IL}\mathcal{S}\,. \tag{3.2}$$

The operators $\mathcal{R}^{IL}$ form a representation of the transverse SO(9) algebra and are transverse angular momentum operators. Extensive use will be made later of the SO(9) Fierz identity

$$\mathcal{S}^A\mathcal{S}^B = \frac{1}{2}\delta^{AB} + \frac{1}{32}(\gamma_{IJ})^{AB}\mathcal{S}\gamma^{IJ}\mathcal{S} + \frac{1}{96}(\gamma_{IJK})^{AB}\mathcal{S}\gamma^{IJK}\mathcal{S}\,, \tag{3.3}$$

which is valid for sixteen-component Majorana spinors.

3.1 Summary of vertex operators

Graviton vertex operator: the transverse graviton vertex operator is given by

$$U_h = h_{IJ}\left(\dot X^I \dot X^J - 2\dot X^I \mathcal{R}^{JM}k_M + 2\mathcal{R}^{IL}\mathcal{R}^{JM}k_L k_M\right), \tag{3.4}$$

while the longitudinal vertex operators are given by

$$U_{h^-} = -h_I{}^- p^+\left(\dot X^I - \mathcal{R}^{JM}k_M\right), \tag{3.5}$$
$$U_{h^{--}} = h^{--}p^+p^+\,. \tag{3.6}$$

The graviton wavefunctions in these expressions satisfy the conditions (2.18).

Gravitino vertex operator: The transverse gravitino vertex operator is given by

$$U_\psi = \psi_I\sqrt{p^+}\mathcal{S}\left(\dot X^I - 2\mathcal{R}^{IJ}k_J\right) + \\ + \frac{1}{\sqrt{p^+}}\tilde\psi_I\left\{\gamma\cdot\dot X\mathcal{S}\left(\dot X^I - 2\mathcal{R}^{IJ}k_J\right) + \frac{8}{9}\gamma^L\mathcal{S}\mathcal{R}^{IJ}\mathcal{R}^{LM}k_Jk_M\right\}, \tag{3.7}$$

while the longitudinal gravitino vertex is given by

$$U_{\psi^-} = -\psi^- p^+\mathcal{S} - \tilde\psi^- p^+\gamma\cdot\dot X\mathcal{S}\,. \tag{3.8}$$

They satisfy the physical state conditions (2.20).

Three-form potential vertex operator: The vertex operator for the transverse components of the field strength of the potential, $C^{(3)}$, is given by

$$U_{C^{(3)}} = F_{ILMN} \left(\dot{X}^I - \frac{2}{3} \mathcal{R}^{IJ} k_J \right) \mathcal{R}^{LMN} . \tag{3.9}$$

The longitudinal components of the field strength have vertex operators defined by

$$U_{C^{(3)-}} = -F_{LMN}{}^{-} p^+ R^{LMN} , \tag{3.10}$$

and the components of F are constrained by (2.16).

One direct way of checking the expressions for the vertex operators is to evaluate their matrix elements between on-shell states and compare with the expressions for the three-particle couplings in the field theory. As an example, consider the three-graviton vertex in light-cone gravity which is given by [13]

$$L_3 \sim 2h^{JL} \partial_K h_{IJ} \partial_L h^{IK} - h^{KL} \partial_K h_{IJ} \partial_L h_{IJ} . \tag{3.11}$$

This is easily seen to be equal to the expression obtained by taking the a matrix element of the graviton vertex operators,

$$A_3 = \langle h_1, k_1 | h_2^{IJ} \left(p^I p^J - 2p^I \mathcal{R}^{JM} k_2^M + 2\mathcal{R}^{IL} \mathcal{R}^{JM} k_2^L k_2^M \right) e^{-ik_2 \cdot X} | h_3, k_3 \rangle , \tag{3.12}$$

where $|h_3, k_3\rangle = h_3^{IJ} e^{-ik_3 \cdot X} |IJ\rangle$ and use has been made of the fact that $\mathcal{R}^{JM}$ acts as an angular momentum on the graviton states so that,

$$\mathcal{R}^{JM} |RS\rangle = \frac{1}{2} \left(\delta^{MR} |JS\rangle - \delta^{JR} |MS\rangle + \delta^{MS} |RJ\rangle - \delta^{JS} |RM\rangle \right) . \tag{3.13}$$

3.2 Supersymmetry transformation on vertex operators

The vertex operators defined in (3.4) to (3.9) may be derived from the requirement that they form a representation of the linearized supersymmetry transformations of eleven-dimensional supergravity. We will go some way to demonstrating this explicitly although we will not give a complete discussion of all the supersymmetry transformations.

Proceeding by analogy with the method used in light-cone gauge string theory [14] (which was explained more fully in [15]), the 32-component supercharge, $Q^{\hat{A}}$, decomposes into the SO(9) spinors Q^A (with parameter η^A) and $\tilde{Q}^A$ (with parameter ϵ^A). Whereas the momentum and the Q^A supercharge are given by their free-field expressions the operator $\tilde{Q}^A$, as well as the hamiltonian H, receive interaction corrections at every order in perturbation theory. The algebra,

$$\{Q^A, Q^B\} = 2p^+ \delta^{AB} , \qquad \{\tilde{Q}^A, Q^B\} = \sqrt{2} \gamma_I \dot{X}^I , \qquad \{\tilde{Q}^A, \tilde{Q}^B\} = 2H \delta^{AB} ,$$
$$[H, Q^A] = 0 = [H, \tilde{Q}^A] , \tag{3.14}$$

determines the form of the interaction corrections.

We may expand the interaction operators in powers of the fields, $H = H_2 + H_3 + \cdots$, $\tilde{Q}^A = \tilde{Q}^A_2 + \tilde{Q}^A_3 + \cdots$, where a subscript $_n$ indicates an operator that has n powers of the fields. The lowest-order corrections to the free-field algebra are given by

$$[H_3, Q^A] = 0\,, \qquad \{\tilde{Q}^A_3, Q^B\} = 0\,, \tag{3.15}$$

and

$$\begin{aligned} [H_2, \tilde{Q}^A_3] + [H_3, \tilde{Q}^A_2] &= 0\,, \\ \{\tilde{Q}^A_3, \tilde{Q}^B_2\} + \{\tilde{Q}^A_2, \tilde{Q}^B_3\} &= 2H_3\delta^{AB}\,. \end{aligned} \tag{3.16}$$

These expressions determine how the free-field generators (H_2, Q^A and $\tilde{Q}^A_2$) transform the interaction terms.

To make contact with the vertex operator description we can associate each n-field interaction term with a state in the n-particle Hilbert space so that

$$H_3 \to |H\rangle_3\,, \qquad \tilde{Q}^A_3 \to |\tilde{Q}^A\rangle_3\,. \tag{3.17}$$

Cubic interaction vertices are then identified with matrix elements in which one of the three external legs is a physical on-shell state, Φ,

$$V_\Phi = \langle\Phi|H\rangle_3\,, \qquad W^A_\Phi = \langle\Phi|\tilde{Q}^A\rangle_3\,. \tag{3.18}$$

These are representations of the interactions in a two-particle space that can be represented by single creation and annihilation operators in a standard manner. The supersymmetry transformations of the vertices therefore follow by taking the corresponding matrix element of (3.16),

$$\begin{aligned} \delta V_h &= V_\psi(\delta\psi)\,, \\ \delta V_\psi &= V_h(\delta h) + V_{C^{(3)}}(\delta C^{(3)})\,, \\ \delta V_{C^{(3)}} &= V_\psi(\delta\psi)\,, \\ \tilde{\delta} V_h &= V_\psi(\tilde{\delta}\psi) + \epsilon^A \frac{d}{d\tau} W^A_h\,, \\ \tilde{\delta} V_\psi &= V_h(\tilde{\delta} h) + V_{C^{(3)}}(\tilde{\delta} C^{(3)}) + \epsilon^A \frac{d}{d\tau} W^A_\psi\,, \\ \tilde{\delta} V_{C^{(3)}} &= V_\psi(\tilde{\delta}\psi) + \epsilon^A \frac{d}{d\tau} W^A_{C^{(3)}}\,, \end{aligned} \tag{3.19}$$

where the δ's are the variations associated with the free-field supersymmetry operators in (3.15) and (3.16) and the time derivatives are generated by H_2. The operators W_h, W_ψ and $W_{C^{(3)}}$, defined in (3.18), transform under supersymmetry as

$$\delta W_h = W_\psi(\delta\psi)\,, \qquad \delta W_\psi = W_h(\delta h) + W_{C^{(3)}}(\delta C^{(3)})\,, \qquad \delta W_{C^{(3)}} = W_\psi(\delta\psi)\,, \tag{3.20}$$

$$\begin{aligned}\epsilon_1^A\tilde{\delta}_{\epsilon_2}W_h^A - \epsilon_1^A W_\psi^A(\tilde{\delta}_{\epsilon_2}\psi) - (1\leftrightarrow 2) &= 2\epsilon_1^A\epsilon_2^A V_h, \\ \epsilon_1^A\tilde{\delta}_{\epsilon_2}W_\psi^A - \epsilon_1^A\big(W_h^A(\tilde{\delta}_{\epsilon_2}h) + W_{C^{(3)}}^A(\tilde{\delta}_{\epsilon_2}C^{(3)})\big) - (1\leftrightarrow 2) &= 2\epsilon_1^A\epsilon_2^A V_\psi\,, \\ \epsilon_1^A\tilde{\delta}_{\epsilon_2}W_{C^{(3)}}^A - \epsilon_1^A W_\psi^A(\tilde{\delta}_{\epsilon_2}\psi) - (1\leftrightarrow 2) &= 2\epsilon_1^A\epsilon_2^A V_C^{(3)}\,. \end{aligned} \tag{3.21}$$

In (3.19) and (3.21) the supersymmetry-transformed wavefunctions such as δh are given by the linearized supersymmetry transformations of the fields, (2.26), (2.31) and (2.32).

3.3 Linear supersymmetry on the graviton vertex

Under the linearly realized supersymmetry transformation, $\delta X^I = 0$, $\delta\mathcal{S} = \sqrt{p^+}\eta$, the graviton vertex operator, $V_h = U_h e^{ik\cdot X}$ (with U_h defined in (3.4)), transforms according to

$$\begin{aligned}\delta V_h &= -h_{IJ}\sqrt{p^+}\left(\dot{X}^I\eta\gamma^{JK}\mathcal{S}\;k_K + \frac{1}{2}\eta\gamma^{IK}\mathcal{S}\;\mathcal{S}\gamma^{JL}\mathcal{S}\;k_K k_L\right)e^{-ik\cdot X}\,, \\ &= k_{[L}h_{I]J}\eta\gamma^{LI}\sqrt{p^+}\mathcal{S}\big(\dot{X}^J - 2\mathcal{R}^{JM}k_M\big)e^{-ik\cdot X}\,. \end{aligned} \tag{3.22}$$

Hence the linear supersymmetry transformation of the graviton vertex operator is the gravitino vertex, $U_\psi e^{-ik\cdot X}$ (with U_ψ defined by (3.7)), with the wavefunction $\delta\psi$ given by (2.31). None of the other terms in the gravitino vertex (3.7) contribute since they depend on $\delta\tilde{\psi}$ (2.31), which vanishes for the linear components of supersymmetry when $k^+ = 0$.

3.4 Nonlinear supersymmetry on the graviton vertex

To determine the full structure of gravitino vertex operator including the terms in (3.7) depending on $\tilde{\psi}$, a non-linearly realized supersymmetry transformation on U_h has to be performed. Under the non-linear supersymmetry transformations $\tilde{\delta}X^I = -2\epsilon\gamma^I\mathcal{S}/\sqrt{p^+}$, $\tilde{\delta}\mathcal{S} = i\dot{X}^I\gamma^I\epsilon/\sqrt{p^+}$. The transformation of the graviton vertex is given by

$$\begin{aligned}\sqrt{p^+}\tilde{\delta}V_h &= h_{IJ}\left(-\dot{X}^I\;(\tilde{\delta}\mathcal{S})\gamma^{JK}\mathcal{S}\;k_K + \frac{1}{2}(\tilde{\delta}\mathcal{S})\gamma^{IK}\mathcal{S}\;\mathcal{S}\gamma^{JL}\mathcal{S}\;k_K k_L\right)e^{-ik\cdot X} + \\ &\quad + i\epsilon\gamma^I k_I\mathcal{S}\,U_h e^{-ik\cdot X} \\ &= -ih_{IJ}\epsilon\left(\gamma^{IL}\gamma^N - 2\delta^{NL}\gamma^I + 2\delta^{NI}\gamma^L\right)\mathcal{S}k_L\dot{X}^N\left(\dot{X}^J - 2\mathcal{R}^{JM}k_M\right)e^{-ik\cdot X} + \\ &\quad + 2ih_{IJ}\epsilon\gamma^N k_N\mathcal{S}\left(\dot{X}^I\dot{X}^J - 2\dot{X}^I\mathcal{R}^{JM}k_M + 2\mathcal{R}^{IL}\mathcal{R}^{JM}k_L k_M\right)e^{-ik\cdot X}\,. \end{aligned} \tag{3.23}$$

The last term, which is of order $\mathcal{S}^5$, can be re-expressed using (3.3) as

$$ih_{IJ}\epsilon\gamma^N k_N\mathcal{S}\;\mathcal{R}^{IL}\mathcal{R}^{JM}k_L k_M = -\frac{2}{9}ih_{IJ}\epsilon\gamma^{IL}\gamma^K\mathcal{S}\;\mathcal{R}^{KM}\mathcal{R}^{JN}k_L k_M k_N\,. \tag{3.24}$$

Hence, (3.23) can be written as

$$
\begin{aligned}
\tilde{\delta} V_h &= 2ik^- h_{IJ}\epsilon\gamma^I \sqrt{p^+}\mathcal{S}\big(\dot{X}^J - 2\mathcal{R}^{JM}k_M\big)e^{-ik\cdot X} \\
&\quad + ik_{[L}h_{I]J}\epsilon\gamma^{LI}\frac{1}{\sqrt{p^+}}\left(\gamma\cdot\dot{X}\mathcal{S}(\dot{X}^J - 2\mathcal{R}^{JM}k_M) + \frac{8}{9}\gamma^K\mathcal{S}\mathcal{R}^{KM}\mathcal{R}^{JN}k_M k_N\right)e^{-ik\cdot X} \\
&\quad + \epsilon^A\frac{dW_h^A}{d\tau}\,,
\end{aligned}
\tag{3.25}
$$

where

$$
W_h^A = \frac{1}{\sqrt{p^+}}h_{IJ}(\gamma^I\mathcal{S})^A\big(\dot{X}^J - 2R^{JM}k_M\big)e^{-ik\cdot X}\,. \tag{3.26}
$$

This is precisely of the expected form. The right-hand side of (3.25) is the sum of the gravitino vertex operator (3.7) with wave-function $\tilde{\delta}\psi(\tilde{\psi})$ and the time derivative of W_h, which is thus determined.

3.5 Linear supersymmetry on the gravitino vertex

We shall now consider the linear supersymmetry transformation of the gravitino vertex operator (3.7) which contains much information concerning the graviton and three-form potential vertex operators. In fact, since the transformation contains terms of order $\mathcal{S}^0, \mathcal{S}^2$ and $\mathcal{S}^4$, which are the same orders as in U_h (3.4) and $U_{C^{(3)}}$ (3.9), it will be sufficient to calculate the linear supersymmetry transformation to confirm the structure of the other two vertex operators. The linear supersymmetry transformation, (2.11), of the gravitino vertex is determined by,

$$
\begin{aligned}
\delta V_\psi &= \left(\psi_I p^+\eta + \tilde{\psi}_I\gamma_L\eta\dot{X}^L\right)\left(\dot{X}^I - 2\mathcal{R}^{IJ}k_J\right)e^{-ik\cdot X} \\
&\quad + \left(\psi_I p^+\mathcal{S} + \tilde{\psi}_I\gamma_L\mathcal{S}\dot{X}^L\right)(-\eta\gamma^{IJ}\mathcal{S}k_J)e^{-ik\cdot X} \\
&\quad + \frac{8}{9}\tilde{\psi}_I\gamma_L\eta\mathcal{R}^{IJ}\mathcal{R}^{LM}k_J k_M e^{-ik\cdot X} + \frac{8}{9}\tilde{\psi}_{(I}\gamma_{L)}\mathcal{S}\eta\gamma^{LM}\mathcal{S}\mathcal{R}^{IJ}k_J k_M e^{-ik\cdot X}\,.
\end{aligned}
\tag{3.27}
$$

In order to check that this agrees with the expected transformation it is necessary to use Fierz transformations to rearrange these terms so that η contracts into the gravitino wavefunction ψ or $\tilde{\psi}$. We will make use of four identities, ignoring the longitudinal polarizations, that follow from (3.3). Firstly,

$$
-\psi_I\mathcal{S}\mathcal{S}\gamma^{IJ}\eta k_J = \psi_I\eta\mathcal{R}^{IJ}k_J - 3\psi_I\gamma^L\eta\mathcal{R}^{ILM}k_M\,. \tag{3.28}
$$

Secondly,

$$
\begin{aligned}
-\tilde{\psi}_I\gamma^P\mathcal{S}\mathcal{S}\gamma^{IJ}\eta\dot{X}^P k_J e^{-ik\cdot X} = {} & -(\tilde{\psi}_I\gamma^L\eta - \tilde{\psi}_L\gamma^I\eta)\dot{X}^I\mathcal{R}^{LM}k_M e^{-ik\cdot X} - \\
& - \tilde{\psi}_L\gamma^M\eta k^- p^+\mathcal{R}^{LM}e^{-ik\cdot X} + \\
& + \frac{3}{2}(\tilde{\psi}_I\gamma^{LM}\eta k_N e^{-ik\cdot X} - 2\tilde{\psi}_L\gamma^{IM}\eta k_N)\dot{X}^I\mathcal{R}^{LMN}e^{-ik\cdot X} - \\
& - \frac{3}{2}\tilde{\psi}_L\gamma^{MN}\eta k^- p^+\mathcal{R}^{LMN}e^{-ik\cdot X} + \\
& + 3\tilde{\psi}_L\eta k_M\dot{X}^N\mathcal{R}^{LMN}e^{-ik\cdot X} + \\
& + \frac{d}{d\tau}\left\{\left(i\tilde{\psi}_L\gamma^M\eta\mathcal{R}^{LM} + \frac{3}{2}i\tilde{\psi}_L\gamma^{MN}\eta\mathcal{R}^{LMN}\right)e^{-ik\cdot X}\right\}.
\end{aligned}
\tag{3.29}
$$

The third Fierz identity that we shall use is

$$
\tilde{\psi}_I\gamma^L\mathcal{S}\eta\gamma^{LM}\mathcal{S}\mathcal{R}^{IJ}k_J k_M = \frac{5}{2}\,\tilde{\psi}_I\gamma^L\eta\mathcal{R}^{IJ}\mathcal{R}^{LM}k_J k_M - \frac{9}{4}\,\tilde{\psi}_I\gamma^{LM}\eta k_N\mathcal{R}^{IJ}\mathcal{R}^{LMN}k_J\,,
\tag{3.30}
$$

while the fourth is

$$
\begin{aligned}
\tilde{\psi}_L\gamma^I\mathcal{S}\mathcal{S}\gamma^{LM}\eta\mathcal{R}^{IJ}k_J k_M = {} & -\frac{3}{2}\left(\tilde{\psi}_I\gamma^{LM}\eta k_N - 2\tilde{\psi}_L\gamma^{IM}\eta k_N\right)\mathcal{R}^{LMN}\mathcal{R}^{IJ}k_J + \\
& + 3\,\tilde{\psi}_L\eta k_N\mathcal{R}^{LMN}\mathcal{R}^{MJ}k_J\,.
\end{aligned}
\tag{3.31}
$$

Substituting these identities into (3.27) gives

$$
\begin{aligned}
\delta V_\psi = {} & \tilde{\psi}_{(I}\gamma_{J)}\eta(\dot{X}^I\dot{X}^J - 2\dot{X}^I\mathcal{R}^{JM}k_M + 2\mathcal{R}^{IL}\mathcal{R}^{JM}k_L k_M)e^{-ik\cdot X} + \\
& + \frac{3}{2}\left(\tilde{\psi}_I\gamma^{LM}\eta k_N - 2\tilde{\psi}_L\gamma^{IM}\eta k_N\right)\dot{X}^I\mathcal{R}^{LMN}e^{-ik\cdot X} + \\
& + \frac{d}{d\tau}\left(\frac{3}{2}i\tilde{\psi}_L\gamma^{MN}\eta\mathcal{R}^{LMN}e^{-ik\cdot X}\right) - \\
& - \left(\frac{5}{3}\tilde{\psi}_I\gamma^{LM}\eta k_N - \frac{4}{3}\tilde{\psi}_L\gamma^{IM}\eta k_N\right)\mathcal{R}^{LMN}\mathcal{R}^{IJ}k_J e^{-ik\cdot X} + \delta V'\,,
\end{aligned}
\tag{3.32}
$$

where $\delta V'$ denotes terms which contain non-transverse polarizations.

The first line of (3.32) is the graviton vertex operator (3.4) with a polarization tensor $\delta h_{IJ}(\eta)$ which was defined in (2.26). The terms in the second and third lines of (3.32) have the right form to reproduce the three-form potential vertex operator (3.9) as follows. Firstly using (3.28) and (3.29),

$$
\begin{aligned}
(\tilde{\psi}_I\gamma_{LM}\eta k_N - 2\tilde{\psi}_L\gamma_{IM}\eta k_N)\dot{X}^I\mathcal{R}^{LMN}e^{-ik\cdot X} = {} & 4\tilde{\psi}_{[I}\gamma_{LM}\eta k_{N]}\dot{X}^I\mathcal{R}^{LMN}e^{-ik\cdot X} - \\
& - i\frac{d}{d\tau}\left(\tilde{\psi}_L\gamma_{MN}\eta\mathcal{R}^{LMN}e^{-ik\cdot X}\right).
\end{aligned}
\tag{3.33}
$$

Furthermore, using (3.30) and (3.31), the terms in the third line of (3.32) can be written as

$$
\begin{aligned}
& -\left(\frac{5}{3}\tilde{\psi}_I\gamma_{LM}\eta k_N - \frac{4}{3}\tilde{\psi}_L\gamma_{IM}\eta k_N\right)\mathcal{R}^{LMN}\mathcal{R}^{IJ}k_J e^{-ik\cdot X} = \\
& \qquad = -4\tilde{\psi}_{[I}\gamma_{LM}\eta k_{N]}\mathcal{R}^{LMN}\mathcal{R}^{IJ}k_J e^{-ik\cdot X}\,.
\end{aligned}
\tag{3.34}
$$

The terms in the second and third lines of (3.32) therefore combine into

$$4 \times \frac{3}{2}\tilde{\psi}_{[I}\gamma_{LM}\eta k_{N]}\left(\dot{X}^I - \frac{2}{3}\mathcal{R}^{IJ}k_J\right)\mathcal{R}^{LMN}e^{-ik\cdot X}\,, \tag{3.35}$$

which is the $C^{(3)}$ vertex operator evaluated with the supersymmetry transformed wavefunction.

3.6 Linear supersymmetry on the three-form potential vertex

The structure of the three-form pontential vertex operator has already been determined from the linear supersymmetry transformation of the gravitino vertex but an additional check is given by the linear supersymmetry transformation of the $C^{(3)}$ vertex. Given the expression for $U_{C^{(3)}}$ in (3.9) this vertex can be written as

$$\begin{aligned} V_{C^{(3)}} &= F_{ILMN}\left(\dot{X}^I - \frac{2}{3}\mathcal{R}^{IJ}k_J\right)\mathcal{R}^{LMN}e^{-ik\cdot X} \\ &= -3k_L C^{(3)}_{IMN}\left(\dot{X}^I - \frac{2}{3}\mathcal{R}^{IJ}k_J\right)\mathcal{R}^{LMN}e^{-ik\cdot X} + (\text{total derivative})\,. \end{aligned} \tag{3.36}$$

In the following we will drop the total derivative term, although this is an important contribution to contact interactions. The linear supersymmetry transformation of (3.36) gives

$$\delta U_{C^{(3)}} = -\frac{\sqrt{p^+}}{2}k_L C^{(3)}_{IMN}\eta\gamma^{LMN}\mathcal{S}\left(\dot{X}^I - \frac{2}{3}\mathcal{R}^{IJ}k_J\right) + \frac{\sqrt{p^+}}{12}k_J k_L C^{(3)}_{IMN}\eta\gamma^{IJ}\mathcal{S}\mathcal{S}\gamma^{LMN}\mathcal{S}\,. \tag{3.37}$$

Using the following Fierz identity

$$k_J k_L C^{(3)}_{IMN}\eta\gamma^{IJ}\mathcal{S}\mathcal{S}\gamma^{LMN}\mathcal{S} = k_J k_L C^{(3)}_{IMN}\left(\eta\gamma^{LMN}\mathcal{S}\mathcal{S}\gamma^{IJ}\mathcal{S} + \frac{1}{3}\mathcal{S}\gamma^{LMNIP}\eta\mathcal{S}\gamma^{PJ}\mathcal{S}\right), \tag{3.38}$$

eq. (3.37) can be written as

$$\delta U_{C^{(3)}} = -2\sqrt{p^+}k_L C^{(3)}_{IMN}\left(\frac{1}{4}\eta\gamma^{LMN}\mathcal{S}\dot{X}^I - \frac{1}{3}\eta\gamma^{LMN}\mathcal{S}\mathcal{R}^{IJ}k_J + \frac{1}{18}\eta\gamma^{LMNIP}\mathcal{S}\mathcal{R}^{PJ}k_J\right). \tag{3.39}$$

It is easy to check that this expression is equal to the part of $U_\psi(\delta\psi)$ which depends on $C^{(3)}$. Therefore, $\delta U_{C^{(3)}}$ is consistent with the expected supersymmetry transformation (3.19) (ignoring the total derivatives).

4. Dimensional reduction on S^1

Type IIA supergravity arises by dimensional reduction of eleven dimensional supergravity on a circle [16] so that compactification of the vertex operators on a circle of radius R_{11} should coincide with the vertex operators of type IIA supergravity. For simplicity, we will set the momentum in the eleventh dimension to zero so that $k_{11} = 0$. The case of non-zero k_{11} describes interactions of D-particles. In making this reduction we will decompose SO(9) spinors into their SO(8) components so that

$$\mathcal{S}^A = (S^a, \tilde{S}^{\dot{a}}) , \tag{4.1}$$

where $a, \dot{a}$ label the two inequivalent SO(8) spinors ($a, \dot{a} = 1, \ldots, 8$) and $\{S^a, S^b\} = \delta^{ab}$,$\{S^{\dot{a}}, S^{\dot{b}}\} = \delta^{\dot{a}\dot{b}}$. The 16×16 SO(9) gamma matrices decompose into the standard 8×8 matrices, $\gamma^i_{a\dot{b}}$ and $\gamma^i_{\dot{a}b}$, in the following manner,

$$\gamma^I = \begin{pmatrix} 0 & \gamma^i \\ \gamma^i & 0 \end{pmatrix} , \tag{4.2}$$

for $I = 1, \ldots, 8$, and

$$\gamma^{11} \equiv \prod_{I=1}^{8} \gamma^I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{4.3}$$

The following identities hold when $I, J, K = 1, \ldots, 8$,

$$\mathcal{R}^{ij} = \frac{1}{4} S\gamma^{ij} S + \frac{1}{4} \tilde{S}\gamma^{ij}\tilde{S} , \qquad \mathcal{R}^{ijk} = \frac{1}{6} S\gamma^{ijk}\tilde{S} , \tag{4.4}$$

where i, j, k are SO(8) vector indices, while if one of the transverse vector indices is in the eleventh direction we have

$$\begin{aligned} \mathcal{R}^{ij\,11} &= \frac{1}{12} S\gamma^{ij} S - \frac{1}{12} \tilde{S}\gamma^{ij}\tilde{S} , \\ \mathcal{R}^{i\,11} &= \frac{1}{4} \tilde{S}\gamma^i S - \frac{1}{4} S\gamma^i \tilde{S} = \frac{1}{2} \tilde{S}\gamma^i S . \end{aligned} \tag{4.5}$$

The eleven-dimensional Fierz transformation (3.3), gives the following well-known ten-dimensional relations,

$$S^a S^b = \frac{1}{2}\delta^{ab} + \frac{1}{16}\gamma^{ab}_{ij} S\gamma^{ij} S, \qquad \tilde{S}^{\dot{a}}\tilde{S}^{\dot{b}} = \frac{1}{2}\delta^{\dot{a}\dot{b}} + \frac{1}{16}\gamma^{\dot{a}\dot{b}}_{ij}\tilde{S}\gamma^{ij}\tilde{S} . \tag{4.6}$$

To begin with we will review how the IIA supergravity vertices arise from the point-particle limit of those of the IIA superstring.

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4.1 Point particle limit of the IIA superstring vertices

The vertex operator for any massless field, Φ, in IIA superstring theory has the form

$$V_{\Phi}^{(IIA)} = U_{\Phi}^{(IIA)} e^{-ik\cdot X}\,, \tag{4.7}$$

where

$$U_{\Phi}^{(IIA)} = \zeta^{\Phi}_{\mathcal{A}\mathcal{B}} \mathcal{O}^{\mathcal{A}} \tilde{\mathcal{O}}^{\mathcal{B}}\,, \tag{4.8}$$

and $\zeta^{\Phi}_{\mathcal{A}\mathcal{B}}$ is the on-shell wavefunction for Φ which is either a second-rank tensor, a spinor-vector or a bi-spinor — $\mathcal{A}$ and $\mathcal{B}$ may either be vector or spinor labels. The prefactor is correspondingly a product of left-moving and right-moving bosonic or fermionic operators. In the light-cone gauge these are classified into SO(8) representations.

The left-moving and right-moving bosonic operators are either transverse vectors,

$$\mathcal{B}^i = \partial X^i - \frac{1}{2} S\gamma^{il} S k_l\,, \qquad \tilde{\mathcal{B}}^i = \bar{\partial} X^i - \frac{1}{2}\tilde{S}\gamma^{il}\tilde{S} k_l\,, \tag{4.9}$$

where $i = 1, \ldots, 8$, or singlets,

$$\mathcal{B}^+ = \tilde{\mathcal{B}}^+ = p^+\,. \tag{4.10}$$

The left-moving fermionic prefactor has two pieces corresponding to the two inequivalent SO(8) spinors,

$$\mathcal{F}^a = \sqrt{p^+} S^a\,, \qquad \mathcal{F}^{\dot{a}} = \frac{1}{\sqrt{p^+}} (\gamma\cdot\partial X S)^{\dot{a}} - \frac{1}{6\sqrt{p^+}} : (\gamma^l S)^{\dot{a}} S\gamma^{lm} S : k_m\,, \tag{4.11}$$

while the pieces of the right-moving fermion prefactor are

$$\tilde{\mathcal{F}}^{\dot{a}} = \sqrt{p^+}\tilde{S}^{\dot{a}}\,, \qquad \tilde{\mathcal{F}}^{a} = \frac{1}{\sqrt{p^+}} (\gamma\cdot\bar{\partial} X \tilde{S})^{a} - \frac{1}{6\sqrt{p^+}} : (\gamma^l \tilde{S})^{a} \tilde{S}\gamma^{lm} \tilde{S} : k_m\,. \tag{4.12}$$

In these expressions $\partial X \equiv (\partial_\tau + i\partial_\sigma)X$, $\bar{\partial} X \equiv (\partial_\tau - i\partial_\sigma)X$ and $\{\tilde{S}^{\dot{a}}, S^b\} = 0$. The point-particle limit of IIA string theory is simply obtained by dropping all the σ dependence of the variables so that

$$\partial X^i = \bar{\partial} X^i = \dot{X}^i\,, \tag{4.13}$$

and only the zero modes of S and $\tilde{S}$ are retained.

The tensor wavefunctions ζ_{ij}, $\zeta_i{}^-$ and ζ^{--} describe the massless bosonic fields in the $NS \otimes NS$ sector while the bispinors $\zeta_{ab}\,, \zeta_{\dot{a}\dot{b}}\,, \zeta_{\dot{a}b}$ and $\zeta_{a\dot{b}}$ describe the massless bosons of the $R \otimes R$ sector (where the first index labels the left-movers and the second index labels the right-movers). Using the physical state conditions the latter can be written as,

$$\begin{aligned}
\zeta_{ab} &= F_{ij}\gamma^{ij}_{ab} + F_{ijkl}\gamma^{ijkl}_{ab}\,, \\
\zeta_{\dot{a}\dot{b}} &= F_{ij}\gamma^{ij}_{\dot{a}\dot{b}} + F_{ijkl}\gamma^{ijkl}_{\dot{a}\dot{b}}\,, &(4.14)\\
\zeta_{a\dot{b}} &= F_{i}{}^{-}\gamma^{i}_{a\dot{b}} + F_{ijk}{}^{-}\gamma^{ijk}_{a\dot{b}}\,, &(4.15)\\
\zeta_{\dot{a}b} &= F_{i}{}^{-}\gamma^{i}_{\dot{a}b} + F_{ijk}{}^{-}\gamma^{ijk}_{\dot{a}b}\,, &(4.16)
\end{aligned}$$

where $F_i{}^- = k_i C^{(1)-} - k^- C_i^{(1)}$, $F_{ij} = 2k_{[i}C_{j]}^{(1)}$ and $F_i{}^+ = 0$ while $F_{ijk}{}^-$ and F_{ijkl} are defined similarly in terms of $C^{(3)}$. The fermions are described by the spinor-vectors ζ_{ia}, $\zeta_{i\dot{a}}$, $\zeta_a{}^-$ and $\zeta_{\dot{a}}{}^-$.

4.2 Reduction of the graviton vertex operator

The reduction of the eleven-dimensional graviton vertex operator (3.4) on a circle gives rise to the vertex operators of the graviton (h), dilaton (ϕ) and one-form gauge potential ($C^{(1)}$) in IIA supergravity theory. For simplicity, we will here consider only the transverse parts of the vertex operators.

The covariant relation between the eleven-dimensional M-theory metric and the ten-dimensional IIA metric in the string frame is given by the ansatz [2] (setting the $R \otimes R$ vector field to zero),

$$G_{\hat{\mu}\hat{\nu}}dx^{\hat{\mu}}dx^{\hat{\nu}} = e^{4/3\phi}(dx_{11})^2 + e^{-2/3\phi}g^{\mathrm{IIA}}_{\mu\nu}dx^{\mu}dx^{\nu} \tag{4.17}$$

from which the corresponding metric fluctuations are related by

$$h_{\mu\nu} = h^{\mathrm{IIA}}_{\mu\nu} - \frac{2}{3}\phi\,\eta_{\mu\nu}, \qquad h_{\mathbb{1}\mathbb{1}} = \frac{4}{3}\phi\,, \tag{4.18}$$

where $\mu,\nu = 0,1,\ldots,9$. However, we want to compare the ten-dimensional and eleven-dimensional theories in their respective light-cone gauges. In particular, setting $h_{+-} = 0$ in (4.18) leads to

$$h^{\mathrm{IIA}}_{+-} = -\frac{2}{3}\phi = -\frac{1}{2}h_{\mathbb{1}\mathbb{1}}\,, \tag{4.19}$$

while the usual ten-dimensional light-cone gauge condition in the type IIA theory sets $h^{\mathrm{IIA}}_{+-} = 0$. This means that in order to compare with the usual light-cone vertices of the IIA theory it is necessary to perform a gauge transformation in order to transform away the component h^{IIA}_{+-} after the identification (4.18) is made. But the condition $h^{\mathrm{IIA}}_{+-} = 0$ can obviously only be compatible with the eleven-dimensional condition $h_{+-} = 0$ by relaxing the tracelessness condition, $h^I_I = h^i_i + h_{11\,11} = 0$. As remarked earlier, in the kinematic regime $k^+ = 0$ this tracelessness condition is not a necessary consequence of the choice of light-cone gauge, and we will choose the very convenient alternative constraint

$$h_{11\,11} = 0\,. \tag{4.20}$$

Upon dimensional reduction the ten-dimensional graviton and dilaton vertex operators are both contained in

$$\begin{aligned} U_h^{10} = {} & h_{ij}\left(\dot{X}^i - \frac{1}{2}S\gamma^{il}Sk_l\right)\left(\dot{X}^j - \frac{1}{2}\tilde{S}\gamma^{jm}\tilde{S}k_m\right) + \frac{1}{8}h_{\mathbb{1}\mathbb{1}}S\gamma^{il}S\tilde{S}\gamma^{im}\tilde{S}k_lk_m - \\ & - \frac{1}{24}\left(k^p h_{pl} - \frac{1}{2}k_l h^p_p\right)\left(S\gamma^{il}SS\gamma^{im}S + \tilde{S}\gamma^{il}\tilde{S}\tilde{S}\gamma^{im}\tilde{S}\right)k_m \\ = {} & h_{ij}\left(\dot{X}^i - \frac{1}{2}S\gamma^{il}Sk_l\right)\left(\dot{X}^j - \frac{1}{2}\tilde{S}\gamma^{jm}\tilde{S}k_m\right) \\ & + \frac{1}{8}h_{\mathbb{1}\mathbb{1}}\left(S\gamma^{il}S\tilde{S}\gamma^{im}\tilde{S} - \frac{1}{6}S\gamma^{il}SS\gamma^{im}S - \frac{1}{6}\tilde{S}\gamma^{il}\tilde{S}\tilde{S}\gamma^{im}\tilde{S}\right)k_lk_m\,, \end{aligned} \tag{4.21}$$

where the light-cone de Donder gauge condition (the first condition in (2.16)) has been used. Substituting (4.18) and (4.19) in (4.21) gives the string-frame expression,

$$\begin{aligned} U_h^{\mathrm{IIA}} = {} & h_{ij}^{\mathrm{IIA}}\left(\dot{X}^i - \frac{1}{2}S\gamma^{il}Sk_l\right)\left(\dot{X}^j - \frac{1}{2}\tilde{S}\gamma^{jm}\tilde{S}k_m\right) + \\ & + h_{+-}^{\mathrm{IIA}}\left\{\dot{X}^i\dot{X}_i - \frac{1}{2}\dot{X}_i(S\gamma^{il}S + \tilde{S}\gamma^{il}\tilde{S})k_l + \frac{1}{24}(S\gamma^{il}SS\gamma_i^{\ m}S + \tilde{S}\gamma^{il}\tilde{S}\tilde{S}\gamma_i^{\ m}\tilde{S})k_lk_m\right\}. \end{aligned} \tag{4.22}$$

It is easy to see that the terms in the first line of (4.22) have the form of the point-particle limit of the IIA graviton vertex operator. The dilaton vertex can be identified as the trace part of the h_{ij}^{IIA} vertex. The presence of the term with polarization tensor h_{+-}^{IIA} is simply a reflection of the fact that the unphysical h_{+-}^{IIA} polarization (4.19) is generated by the dimensional reduction (4.17) in the light-cone gauge. Therefore, the condition (4.20) $h_{\mathbb{1}\mathbb{1}} = 0$ is exactly what is needed to remove the redundant terms in (4.22) and obtain the IIA graviton and dilaton vertex operators.

The vertex operator of the one-form gauge potential of the IIA theory is obtained from the graviton vertex operator by the identification $h_{11\mu} = C^{(1)}_\mu$. Taking one of the indices in (3.4) and (3.5) to be 11 gives

$$\begin{aligned} U^{10}_{C^{(1)}} & = h_{i\,11}\left[-\dot{X}^iS\gamma^j\tilde{S}k_j + \frac{1}{4}\left(S\gamma^{il}S + \tilde{S}\gamma^{il}\tilde{S}\right)S\gamma^j\tilde{S}\,k_lk_j\right] \\ & = -k_jh_{i\,11}\left[S\gamma^j\tilde{S}\dot{X}^i - \frac{1}{24}\left(S\gamma^{ji}\gamma^l\tilde{S}\tilde{S}\gamma^{lm}\tilde{S} - \tilde{S}\gamma^{ji}\gamma^lSS\gamma^{lm}S\right)k_m\right], \end{aligned} \tag{4.23}$$

where the Fierz identity for SO(8) spinors

$$\zeta^i\tilde{S}\gamma^mSS\gamma^{il}Sk_lk_m = -\frac{1}{6}\zeta^i\tilde{S}\gamma^{il}\gamma^pSS\gamma^{pq}Sk_lk_q \tag{4.24}$$

has been used. This coincides with the expression for the one-form vertex operator of the IIA theory up to a total derivative which is irrelevant here.

4.3 Reduction of the gravitino vertex operator

The eleven-dimensional gravitino $\Psi_{\hat{\mu}}$ decomposes into two ten-dimensional Majorana-Weyl spinor-vectors $\Psi_{L\,\mu}$, $\Psi_{R\,\mu}$,

$$\Psi_{L\,\mu} = \frac{1}{2}(1+\Gamma^{11})\Psi_{\mu}, \qquad \Psi_{R\,\mu} = \frac{1}{2}(1-\Gamma^{11})\Psi_{\mu}\,, \tag{4.25}$$

where the subscripts L, R correspond to the two chiralities which are correlated in IIA string theory with the left and right directions on the worldsheet.

Upon compactification, the above decomposition is accompanied with a shift of the eleven-dimensional gravitino which is given by the covariant relation,

$$\Psi_{\mu} = \Psi_{\mu}^{\mathrm{IIA}} - \frac{1}{2}\Gamma_{\mu}\Gamma_{11}\Psi_{11}\,, \tag{4.26}$$

where $\Psi_{\mu}^{\mathrm{IIA}}$ is the gravitino wavefunction in the IIA string frame. Given the eleven-dimensional light-cone gauge condition, $\Psi^{+}=0$, the type IIA theory has nonvanishing Ψ^{+}_{IIA} proportional to Ψ_{11}. This is analogous to the earlier discussion in which we saw that in general $h^{\mathrm{IIA}}_{+-}\neq 0$ when $h_{+-}=0$. Just as this was remedied by making the choice $h_{11\,11}=0$ as a metric constraint, it is convenient to make the choice $\Psi_{11}=0$ as an alternative to the eleven-dimensional gamma-tracelessness condition in (2.14). In this way the supersymmetry transformations of section 3 still work with the trace part of the graviton vertex mapped to the gamma-trace part of the gravitino vertex.

The physical state conditions for the gravitino are then

$$k^{\hat{\mu}}k_{\hat{\mu}} = 0\,, \qquad k^{\hat{\mu}}\Psi_{\hat{\mu}} = 0\,, \qquad k\cdot\Gamma\Psi_{\hat{\mu}} = k_{\hat{\mu}}\Gamma\cdot\Psi\,, \qquad \Psi_{1\!\!1} = 0\,, \tag{4.27}$$

which can be rewritten as, in the light-cone gauge with $k^{+}=0$,

$$\begin{aligned} &k^{I}k_{I} = 0\,, \qquad k^{I}\psi_{I}(\tilde{\psi}_{I}) = 0\,, \qquad \psi_{1\!\!1} = 0 = \tilde{\psi}_{1\!\!1} \\ &k^{I}\gamma_{I}\tilde{\psi}_{\hat{\mu}} = k_{\hat{\mu}}\gamma^{I}\tilde{\psi}_{I}\,, \qquad k^{I}\gamma_{I}\psi_{\hat{\mu}} - k_{\hat{\mu}}\gamma^{I}\psi_{I} = \sqrt{2}(k^{-}\tilde{\psi}_{\hat{\mu}} - k_{\hat{\mu}}\tilde{\psi}^{-})\,. \end{aligned} \tag{4.28}$$

The light-cone components of the spinors, (4.25), are obtained as before by projecting with $P^{\pm}$,

$$\begin{aligned} \psi^{a}_{L\,i} &= (P^{+}\Psi_{L\,i})^{a}\,, \qquad & \tilde{\psi}^{\dot{a}}_{L\,i} &= (P^{-}\Psi_{L\,i})^{\dot{a}}\,, \\ \psi^{\dot{a}}_{R\,i} &= (P^{+}\Psi_{R\,i})^{\dot{a}}\,, \qquad & \tilde{\psi}^{a}_{R\,i} &= (P^{-}\Psi_{R\,i})^{a}\,. \end{aligned} \tag{4.29}$$

Without loss of generality, we will consider only the left-handed gravitino, $\Psi_{L\,\mu}$. The transverse part of the dimensional reduction of the gravitino vertex operator, (3.7), gives

$$\begin{aligned}U^{10}_{\psi_L} = {} & \sqrt{p^+}\psi_i S\left(\dot{X}^i - \frac{1}{2}\tilde{S}\gamma^{ij}\tilde{S}k_j\right) + \\ & + \frac{1}{\sqrt{p^+}}\tilde{\psi}_i\left(\gamma\cdot\dot{X}S - \frac{1}{6}\gamma_l SS\gamma^{lm}k_m\right)\left(\dot{X}^i - \frac{1}{2}\tilde{S}\gamma^{ij}\tilde{S}k_j\right) - \\ & - \frac{1}{2}\sqrt{p^+}\psi_i SS\gamma^{ij}k_j - \frac{1}{9\sqrt{p^+}}\tilde{\psi}_i\tilde{S}\tilde{S}\gamma^{ij}\tilde{S}S\gamma^m\tilde{S}k_jk_m + \\ & + \frac{1}{18\sqrt{p^+}}\tilde{\psi}_i\gamma_l S\left(S\gamma^{ij}SS\gamma^{lm}S + \tilde{S}\gamma^{ij}\tilde{S}\tilde{S}\gamma^{lm}\tilde{S}\right)k_jk_m\,,\end{aligned} \tag{4.30}$$

where the subscript L has been omitted and the following Fierz,

$$\tilde{\psi}_i\gamma_j SS\gamma_{il}Sk_l = \frac{1}{3}\tilde{\psi}_j\gamma_i SS\gamma_{il}Sk_l\,, \tag{4.31}$$

has been used which can be derived using (4.6). The terms in the first line of (4.30) are apparently the vertex operator for the spin-3/2 states in the point-particle limit of IIA string theory from which the gamma-trace part can be separated as the dilatino vertex operator. It would, then, be expected that the remaining terms in (4.30) should vanish. Indeed, using the physical state conditions, (4.28), and the ten-dimensional Fierz, (4.6), it can be shown that these terms vanish up to total derivative terms.

4.4 Reduction of the three-form potential vertex operator

The eleven-dimensional three-form potential gives rise to both the $NS\otimes NS$ two-form, $B_{\mu\nu}$, and the $R\otimes R$ three-form potential in ten dimensions. The components of (3.9) with one index in the eleventh direction results in the $NS\otimes NS$ two-form vertex operator upon compactification,

$$\begin{aligned}U^{10}_B &= \frac{1}{2}F_{\mathbb{1}lmn}\left[-\frac{2}{3}\mathcal{R}^{\mathbb{1}j}\mathcal{R}^{lmn}k_j - 3\left(\dot{X}^l - \frac{2}{3}\mathcal{R}^{lj}k_j\right)\mathcal{R}^{mn\mathbb{1}}\right] \qquad (4.32)\\ &= -\frac{1}{4}F_{\mathbb{1}lmn}\left[\dot{X}^l\left(S\gamma^{mn}S - \tilde{S}\gamma^{mn}\tilde{S}\right) + \frac{1}{3}S\gamma^{lj}S\tilde{S}\gamma^{mn}\tilde{S}k_j - \frac{1}{6}S\gamma^{mn}S\tilde{S}\gamma^{lj}\tilde{S}k_j\right],\end{aligned}$$

where use has been made of (4.4), (4.5) and the Fierz identity

$$k_jF_{\mathbb{1}lmn}S\gamma^j\tilde{S}\tilde{S}\gamma^{lmn}S = \frac{3}{4}k_jF_{\mathbb{1}lmn}S\gamma^{jl}S\tilde{S}\gamma^{mn}\tilde{S}\,. \tag{4.33}$$

Up to a total derivative (4.32) can be written as

$$U^{10}_B = \frac{1}{2}C^{(3)}_{\mathbb{1}lm}\left[\dot{X}^l\left(S\gamma^{mn}S - \tilde{S}\gamma^{mn}\tilde{S}\right)k_n + \frac{1}{2}S\gamma^{lj}S\tilde{S}\gamma^{mn}\tilde{S}k_jk_n\right]. \tag{4.34}$$

Using the identification $C^{(3)}_{\mathbb{1}lm} = B_{lm}$ this expression coincides with the the point particle limit of the IIA $NS\otimes NS$two-form vertex.

The $R\otimes R$ three-form potential vertex is obtained by taking all the indices of (3.9) to be in the transverse SO(8)

$$U^{10}_{C^{(3)}} = \frac{1}{6}F_{ilmn}\left(\dot{X}^i - \frac{1}{6}S\gamma^{ij}Sk_j - \frac{1}{6}\tilde{S}\gamma^{ij}\tilde{S}k_j\right)S\gamma^{lmn}\tilde{S}\,. \tag{4.35}$$

After a few manipulations this is seen to be equivalent to the vertex of IIA supergravity,

$$U^{\mathrm{IIA}}_{C^{(3)}} = F_{ilmn}\left[S\gamma^{ilmn}\gamma^p\tilde{S}\left(\dot{X}^p - \frac{1}{6}\tilde{S}\gamma^{pq}\tilde{S}k_q\right) + \tilde{S}\gamma^{ilmn}\gamma^p S\left(\dot{X}^p - \frac{1}{6}S\gamma^{pq}Sk_q\right)\right]. \tag{4.36}$$

The superscript $^{\mathrm{IIA}}$ indicates that the vertex is that of IIA supergravity (while the superscript ^{10} in (4.35) means the vertex is obtained from dimensional reduction of the appropriate eleven-dimensional vertex).

5. One-loop amplitudes in the light-cone gauge

In this section the first-quantized description of the eleven-dimensional theory developed in the previous sections will be applied to the calculation of one-loop Feynman diagrams by integrating over the world-lines of the circulating particles. This will fill in some of the details of the calculations of the R^4 and λ^{16} effective interactions of the type IIB theory outlined in [3] and [4]. In addition other interactions of the same dimension are obtained from one-loop processes in the eleven-dimensional theory.

The world-line path integral for the one-loop n particle scattering amplitude in eleven dimensional supergravity is given by

$$A_n = \int\frac{dT}{T}\int\mathcal{D}X\int\mathcal{D}S e^{-\int_0^T dt(\frac{1}{2}\dot{X}^2 + iS\dot{S})}\prod_{r=1}^{n}\left(\int dt^{(r)}V^{(r)}(t^{(r)})\right), \tag{5.1}$$

where the vertex operators representing the emission of on-shell particles were defined in section 3. The one-loop amplitude of eleven-dimensional supergravity compactified to $(11-d)$ dimensions on a d-torus, T^d, can be written as

$$A^{(d)}_n = \frac{1}{\mathcal{V}_d}\int\frac{dT}{T}\int d^{11-d}\mathbf{p}\sum_{\{l_I\}}e^{-T(\mathbf{p}^2+G^{(d)IJ}l_Il_J)}\mathrm{Tr}\left\langle\prod_{r=1}^{n}\left(\int dt^{(r)}V^{(r)}(t^{(r)})\right)\right\rangle, \tag{5.2}$$

where $G^{(d)}_{IJ}$ is the metric on T^d, l^I are the Kaluza-Klein momenta, $\mathcal{V}_d$ is the volume of the torus T^d and $\mathbf{p}$ is the loop momentum transverse to the compact directions. The overall factor $1/\mathcal{V}_d$ is the measure for the summation over the Kaluza-Klein momenta and the trace in (5.2) is taken over the fermionic modes S_A. The brackets denote the path ordered expectation value of the vertex operators evaluated with the following Green functions involving the fluctuating quantum fields.

$$\langle X^i(t)X^j(t')\rangle = \delta^{ij}G_B(t,t') = \delta^{ij}\left(\frac{|t-t'|}{2}+\frac{(t-t')^2}{2T}\right)$$
$$\langle \mathcal{S}^A(t)\mathcal{S}^B(t')\rangle = \delta^{AB}G_F(t,t') = \delta^{AB}\left(\frac{1}{2}\mathrm{sgn}(t-t')-\frac{t-t'}{T}\right). \tag{5.3}$$

The universal momentum factors of e^{-ikX} in the vertex operators give contributions to the expectation value of the form

$$\left\langle \prod e^{-ik_iX(t_i)}\right\rangle = e^{-\sum_{i\neq j}k_ik_jG_B(t_i,t_j)}\,. \tag{5.4}$$

In the following example we will only be interested in the leading terms in the low energy expansion, which means that only the loop amplitude with the lowest number of momenta needs to be considered. Hence the contribution (5.4) can be replaced by 1 in this approximation.

The SO(9) fermionic spinor operator $\mathcal{S}$ can be expressed in terms of eight creation and eight annihilation operators which span the space of 256 polarizations of the supergraviton. The only nonzero contributions come from traces with at least sixteen insertions of the operator $\mathcal{S}^A$. We shall focus on a 'protected' class of amplitudes where the path integral over the fermionic world-line variables in which the vertex operators introduce precisely sixteen fermionic zero modes. For this special class of amplitudes there are no contractions involving the Green functions (5.3). The bosonic vertex operators (3.4)–(3.10) contain the two bilinears

$$\mathcal{R}^{LMN} = \frac{1}{12}\mathcal{S}\gamma^{LMN}\mathcal{S}\,, \qquad \mathcal{R}^{IL} = \frac{1}{4}\mathcal{S}\gamma^{IL}\mathcal{S}\,. \tag{5.5}$$

The sixteen factors of S^A in the trace are provided by the eight $\mathcal{R}$'s in any of the protected one-loop amplitudes with external bosons. The trace of sixteen $\mathcal{S}^A$ is given by

$$\mathrm{Tr}\left(\mathcal{S}^{A_1}\cdots\mathcal{S}^{A_{16}}\right) = \epsilon^{A_1\cdots A_{16}}\,, \tag{5.6}$$

from which it is easy to deduce the tensors that arise from matrix elements of the various different combinations of $\mathcal{R}^{LMN}$ and $\mathcal{R}^{IL}$. For example, the tensor t_{16} defined by the trace involving eight $\mathcal{R}^{IL}$ is defined by

$$t_{16}^{I_1I_2\cdots I_{16}} = \epsilon^{A_1\cdots A_{16}}\gamma^{I_1I_2}_{A_1A_2}\gamma^{I_3I_4}_{A_3A_4}\cdots\gamma^{I_{15}I_{16}}_{A_{15}A_{16}}\,. \tag{5.7}$$

Nontrivial relations between the various tensors can be found using Fierz transformations. Amplitudes with external fermions also contain the trace of sixteen factors of $\mathcal{S}^A$ which can be arranged by Fierz tranformations into the trace of products of eight bilinears of the form (5.5).

We therefore see that for the protected processes the trace over the $\mathcal{S}^A$'s simply produces a kinematic factor. The leading term in the low energy expansion of the amplitude can then be obtained by setting the external momenta to zero in the integrand which makes the integration over the proper times $t^{(r)}$ trivial, simply giving a power T^n in the integrand. The resulting expression for the leading term in the low energy expansion of (5.2) for a protected process with n external states has the form

$$A_n^{(d)} = \frac{1}{\mathcal{V}_d}\tilde{K}_n \int_0^\infty \frac{dT}{T} T^{n+d/2-11/2} \sum_{\{l_I\}} e^{G^{(d)IJ} l_I l_J} F(G^{(d)IJ}, l_I)\,, \tag{5.8}$$

where $\tilde{K}_n$ is the kinematic factor for a specific scattering amplitude depending on particle species and $F(G^{(d)IJ}, l_I)$ is a generic function of the torus metric and the Kaluza-Klein momentum. The extra factor of $T^{d/2-11/2}$ comes from the integration over the non-compact loop momentum $\mathbf{p}$.

An important feature of the protected class of loop amplitudes that we are considering is that the contractions between the vertex operators does not generate any factors of the Green functions (5.3), which have short distance singularities. The only contractions are between factors of $e^{ik\cdot X}$. This feature is directly related to the fact that the contact interactions that are present in the complete theory do not, in fact, contribute to these particular processes. This justifies the fact that we have ignored such interactions.

5.1 Compactification on a circle

We will now review the way in which one-loop calculations in compactified eleven-dimensional supergravity compare with known results from string theory. The simplest case is that of compactification on a circle which is equivalent to IIA string theory in ten dimensions [2]. for simplicity only four-point functions involving four bosonic fields in the $NS \otimes NS$ sector (the graviton, dilaton and two-form potential) will be considered.

The dimensionally-reduced vertex operators of relevance are given by (4.21) for the graviton and (4.34) for the two-form potential. A non-vanishing trace over the fermionic zero modes only arises from the part of the vertices of order $S^2\tilde{S}^2$ which are given by

$$V_h = \frac{1}{4} h_{ij} k_k k_l S\gamma^{ik} S\tilde{S}\gamma^{jl}\tilde{S}\,, \qquad V_B = \frac{1}{4} B_{ij} k_k k_l S\gamma^{ik} S\tilde{S}\gamma^{jl}\tilde{S}\,. \tag{5.9}$$

In [3] the four-graviton one-loop amplitude in M-theory compactified on S^1 was calculated and it reproduced the well-known tree and one loop terms,

$$A_{R^4}^{(1)} = C\tilde{K} + 2\tilde{K}\int_0^\infty d\hat{\tau}\hat{\tau}^{1/2} \sum_{\hat{l}_1>0} e^{-\pi\hat{\tau}\hat{l}_1^2 R_{11}^2} = C\tilde{K} + \tilde{K}\zeta(3)\frac{1}{R_{11}^3}\,. \tag{5.10}$$

The value of the constant $C = 2\pi^2/3$ can be determined by connecting it, via T-duality on a further circle, with the IIB theory. Using the relation $R_{11} = g_s^{2/3}$ (where g_s is the type IIA string coupling) the first term on the right-hand side of (5.10) is interpreted as a one-loop term in type IIA string theory whereas the second term corresponds to a tree-level string theory term.

The kinematic factor $\tilde{K}$ takes the form

$$\tilde{K} = t^{i_1\cdots i_8} t_{j_1\cdots j_8} \bar{R}^{j_1 j_2}_{i_1 i_2} \cdots \bar{R}^{j_7 j_8}_{i_7 i_8}\,, \tag{5.11}$$

where the tensor $t^{i_1\cdots i_8}(i_n = 1,\ldots,8)$ is defined in [17] and the linearized Riemann tensor is replaced by a generalized tensor which also contains the two-form potential and the dilaton

$$\bar{R}_{i_1 i_2 j_1 j_2} = k_{i_1} k_{j_1} \left(h_{i_2 j_2} + B_{i_2 j_2} - \frac{1}{4}\delta_{i_2 j_2}\phi \right). \tag{5.12}$$

This is the tensor that can be obtained from the four-point tree-level amplitude of type II string theories including the $NS \otimes NS$ two-form potential and the dilaton [18].

5.2 Compactification on a two torus and type IIB string theory

The compactification of eleven-dimensional supergravity on a two dimensional torus T^2 of volume $\mathcal{V}_2 = R_8 R_{11}$(in the M-theory frame) and complex structure Ω is related to the IIB string theory compactified on a circle of circumference r_B(in the string frame) [19], where the complex structure is identified as the modulus, τ_B, of IIB string theory. The $\mathrm{SL}(2,\mathbb{Z})$ self-duality of IIB string theory is identified with the invariance of M-theory under the group of large diffeomorphisms of T^2.

The correspondence between the parameters of the two theories is [19]

$$\Omega = \tau_B (\equiv C^{(0)} + i e^{-\phi_B})\,, \qquad r_B = \mathcal{V}_2^{-3/4} \Omega_2^{-1/4} = R_8^{-1} R_{11}^{-1/2}\,, \tag{5.13}$$

where $C^{(0)}, \phi_B$ are the IIB $R \otimes R$ scalar and dilaton, respectively, and $\Omega \equiv \Omega_1 + i\Omega_2$. The metric on the torus is given by

$$G^{(2)}_{ij} = \frac{\mathcal{V}_2}{\Omega_2} \begin{pmatrix} |\Omega|^2 & \Omega_1 \\ \Omega_1 & 1 \end{pmatrix}, \qquad i,j = 8, 11\,. \tag{5.14}$$

The zweibein in a special Lorentz gauge can be written as

$$e^a_i = \sqrt{\frac{\mathcal{V}_2}{\Omega_2}} \begin{pmatrix} \Omega_2 & \Omega_1 \\ 0 & 1 \end{pmatrix} \tag{5.15}$$

where $i, a (= 1,2)$ denote the two-dimensional world and tangent space indices respectively. The zweibein parameterizes the coset $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ where the $\mathrm{SL}(2,\mathbb{R})$ acts by matrix multiplication from the left and the local $\mathrm{U}(1)$ acts from the right which can be used to fix the gauge (5.15). Fixing the local $\mathrm{U}(1)$ symmetry leads to the standard nonlinear realization of the $\mathrm{SL}(2,\mathbb{R})$ transformation acting on the complex structure Ω,

$$\Omega \to \frac{a\Omega + b}{c\Omega + d}\,, \tag{5.16}$$

where the coefficients are integers and satisfy $ad - bc = 1$. It is convenient to go to a complex basis in the tangent space, $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$ in which the zweibein now reads

$$e_i^a = i\sqrt{\frac{\mathcal{V}_2}{\Omega_2}}\begin{pmatrix} \bar{\Omega} & -\Omega \\ 1 & -1 \end{pmatrix}, \qquad a = z, \bar{z} \tag{5.17}$$

and the inverse zweibein is

$$e_a^i = \frac{1}{2\sqrt{\mathcal{V}_2 \Omega_2}}\begin{pmatrix} 1 & -\Omega \\ 1 & -\bar{\Omega} \end{pmatrix}, \qquad a = z, \bar{z}\,. \tag{5.18}$$

We will begin by reviewing the one-loop calculations of four-graviton [3] and sixteen-fermion [4] scattering amplitudes. Amplitudes that involve $C^{(3)}$ will also be calculated and may be of interest in relation to the matrix theory calculation of the three-form potential scattering amplitudes [20]. Applying (5.2) to the case of the two torus compactification, the four-graviton one-loop amplitude [3] can be written as

$$A^{(2)}_{R^4} = \frac{1}{\mathcal{V}_2}\tilde{K}_{R^4}\sum_{m,n}\int_0^\infty \frac{dT}{T^{3/2}} e^{-T|m+n\Omega|^2/\mathcal{V}_2\Omega_2}\,, \tag{5.19}$$

where the kinematic factor $\tilde{K}_{R^4}$ is (5.11) with the two-form pontetial and the dilaton set to zero. A double Poisson resummation turns the sum over the Kaluza-Klein charges (m,n) into a sum over the windings $(\hat{m},\hat{n})$ of the loop around the two cycles of T^2 and (5.19) can be expressed as

$$\mathcal{V}_2 A^{(2)}_{R^4} = \frac{2\pi^2}{3}\tilde{K}\mathcal{V}_2 + \tilde{K}\mathcal{V}_2^{-1/2} f^{(0,0)}(\Omega,\bar{\Omega})\,, \tag{5.20}$$

where $f^{(0,0)}(\Omega,\bar{\Omega})$ is the $\mathrm{SL}(2,\mathbb{Z})$-invariant nonholomorphic modular function

$$f^{(0,0)}(\Omega,\bar{\Omega}) = \sum_{(\hat{m},\hat{n})\neq(0,0)} \frac{\Omega_2^{3/2}}{|\hat{m} + \hat{n}\Omega|^3}\,. \tag{5.21}$$

The R^4 term conserves the U(1) charge of classical IIB supergravity and in the decompactification limit $\mathcal{V}_2 \to \infty$ the first term in (5.20) gives a finite R^4 interaction in the eleven-dimensional theory. Other IIB four-point bosonic amplitudes which conserve the U(1) charge and share the structure (5.20) have kinematic factors that can be schematically written as[3]

$$(\partial\hat{F}_5)^4 + (\partial G \partial G^*)^2 + (\partial P \partial P^*)^2 + R^2(\partial G\partial G^* + \partial P \partial P^*) + \partial P\partial P^* \partial G \partial G^* + \cdots\,, \tag{5.22}$$

[3]Such terms were also discussed in [21]

where the ellipses denote additional mixed terms. Here $\hat{F}_5$ is the self-dual five-form field strength and G is a complex linear combination of the $NS \otimes NS$ and $R \otimes R$ three-form field strengths given by

$$G = \frac{1}{\sqrt{2\tau_2}}(\mathrm{d}C^{(2)} - \tau \mathrm{d}B)\,, \tag{5.23}$$

while the Maurer-Cartan form is given by

$$P_\mu = \frac{i}{2}\frac{\partial_\mu \tau}{\tau_2}\,. \tag{5.24}$$

The fields $\hat{F}_5$, G and P have U(1) charges zero, one and two, respectively, while the complex conjugate quantities, denoted G^* and P^*, have U(1) charges of the opposite sign.

Such terms can be systematically described in linearized IIB superspace in terms of a constrained scalar superfield, $\Phi(x,\theta)$, where θ is a sixteen-component complex Grassmann coordinate that transforms as a Weyl spinor of SO(9, 1). After imposing the appropriate constraints Φ has the expansion [22],

$$\Phi = \tau_0 + \Delta\,, \tag{5.25}$$

where τ_0 is the constant value of the complex scalar field and Δ is the fluctuation defined by

$$\begin{aligned}\Delta = {} & (\tau - \tau_0) + i\bar{\theta}^*\lambda + \bar{\theta}^*\Gamma^{\mu\nu\rho}\theta G_{\mu\nu\rho} + i\bar{\theta}^*\Gamma^{\mu\nu\rho}\theta\bar{\theta}^*\Gamma_\mu\partial_\nu\psi_\rho + \\ & + \bar{\theta}^*\Gamma^{\mu\nu\eta}\theta\bar{\theta}^*\Gamma_\eta{}^{\rho\sigma}\theta R_{\mu\nu\rho\sigma} + \bar{\theta}^*\Gamma^{\mu\nu\rho}\theta\bar{\theta}^*\Gamma^{\sigma\eta\omega}\theta\partial_\mu(\hat{F}_5)_{\nu\rho\sigma\eta\omega} + \\ & + \theta^5\partial^2\psi^*_\mu + \bar{\theta}^*\gamma^{\mu\nu\rho}\theta\bar{\theta}^*\gamma^\sigma\theta\bar{\theta}^*\gamma^\omega\theta(\partial_\sigma\partial_\omega G^*_{\mu\nu\rho}) + \theta^7\partial^3\lambda^* + \theta^8\partial^4\tau^*\,,\end{aligned} \tag{5.26}$$

where λ, ψ_μ are the dilatino and the gravitino, respectively. The terms of order θ^5 and higher have been written schematically (except the term of order θ^6 which will be used later). The linearized interactions of the component fields in the limit of weak coupling ($\tau_0 \to \infty$) are contained in [4]

$$S = (\alpha')^3 \mathrm{Re}\int d^{10}x d^{16}\theta F[\Phi]\,. \tag{5.27}$$

The component interactions are obtained by Taylor expanding F in powers of Δ followed by integration over the 16 Grassmann parameters. Of course, this expression does not capture the modular properties of the complete nonlinear interactions.

The R^4 term occurs at order Δ^4 together with the terms in (5.22). For example, $(\partial G\partial G^*)^2$ is obtained by combining two powers of each of the θ^2 and θ^6 terms. The Grassmann integration generates the pattern of contractions between the tensor fields. The terms in (5.22) also arise naturally from the eleven-dimensional perspective with appropriate identifications of the polarization tensors as will be seen shortly.

The sixteen-dilatino term λ^{16} has U(1) charge 24. This interaction was obtained in [4] by a covariant argument using the fact that the zero-momentum eleven-dimensional gravitino vertex can be expressed in terms of the eleven-dimensional supercharge. The same result can be obtained using the light-cone gauge gravitino vertex operator (3.7). The only contributions come from factors in which there are eight vertices from each of the ψ_I and $\tilde{\psi}_I$ terms in (3.7). This follows from the fact that only terms with no net power of p^+ contribute to the momentum integral. Furthermore, there must be a total of sixteen powers of $\mathcal{S}$ so that the only terms in the vertex (3.7) that contribute to the amplitude are

$$U_\psi = \sqrt{p^+}\psi_I \mathcal{S}\dot{X}^I\,, \qquad U_{\tilde{\psi}} = \frac{1}{\sqrt{p^+}}\tilde{\psi}_I \gamma\cdot\dot{X}\mathcal{S}\dot{X}^I\,. \tag{5.28}$$

The complex dilatino field λ is a ten-dimensional Weyl spinor that is identified with a particular polarization state of the eleven-dimensional gravitino compactified on T^2 so its vertex operator follows simply from (5.28) [4]. The amplitude can then be obtained in much the same way as for the four-graviton case using (5.2). The significant new feature is the presence of a complex function of the Kaluza-Klein momentum in front of the exponential

$$\begin{aligned}\mathcal{V}_2 A_{\lambda^{16}} &= \tilde{K}_{\lambda^{16}} \sum_{m,n} \int \frac{dT}{T} T^{23/2} \left(\frac{1}{\sqrt{\mathcal{V}_2\Omega_2}}(m+n\bar{\Omega})\right)^{24} \\ e^{-T|m+n\Omega|^2/\mathcal{V}_2\Omega_2} &= \tilde{K}_{\lambda^{16}} \mathcal{V}_2^{-1/2} f^{(12,-12)}(\Omega,\bar{\Omega})\,.\end{aligned} \tag{5.29}$$

The kinematic factor is $\tilde{K}_{\lambda^{16}}$ is proportional to $\epsilon^{A_1 A_2 \dots A_{16}}\lambda_{A_1}\lambda_{A_2}\dots\lambda_{A_{16}}$ and $f^{(12,-12)}$ is a modular form of weight $(12,-12)$ that is given by

$$f^{(12,-12)}(\Omega,\bar{\Omega}) = \frac{\Gamma(27/2)}{\Gamma(3/2)} \sum_{(\hat{m},\hat{n})\neq(0,0)} \Omega_2^{3/2} \frac{(\hat{m}+\hat{n}\bar{\Omega})^{24}}{|\hat{m}+\hat{n}\Omega|^{27}}\,. \tag{5.30}$$

In contrast to the U(1) charge conserving amplitudes considered before, the λ^{16} term violates 24 units of U(1) charge and is not present in the eleven-dimensional theory. This is seen explicitly from (5.29) which vanishes in the eleven-dimensional limit $\mathcal{V}\to\infty$, while it survives in the ten-dimensional IIB limit. This is true for all the other U(1) charge violating terms. In [23] the explicit form of (5.30) was derived from space-time supersymmetry and $SL(2,\mathbb{Z})$ invariance of IIB string theory alone.

In [4] the scalar field strength P_μ (5.24) of IIB supergravity was identified with the component $\omega_{z\mu z}$ of the spin connection of eleven dimensional supergravity on T^2 which can be written as

$$\omega_{z\mu z} = \frac{1}{2} e_z^i e_z^j \partial_\mu h_{ij}, \tag{5.31}$$

where $i,j = 8, 11$ and P_μ^* corresponds to $\omega_{\bar{z}\mu\bar{z}}$. The one-loop amplitude that generates the interaction $(\partial P \partial P^*)^2$ in (5.22) gets contributions from the factors $\mathcal{R}^{zl}\mathcal{R}^{zm}$ and $\mathcal{R}^{\bar{z}l}\mathcal{R}^{\bar{z}m}$ in the graviton vertex operator (3.4).

We will now consider the vertex operators for the three-form potential $C^{(3)}$ which reduce, upon compactification on a torus, to the IIB two-form potentials and four-form potential $C^{(4)}$. The correspondence is given by

$$C^{(3)}_{\mu\nu\rho} = C^{(4)}_{\mu\nu\rho 8}, \qquad C^{(3)}_{\mu\nu\mathbb{1}} = B_{\mu\nu}, \qquad C^{(3)}_{\mu\nu 8} = C^{(2)}_{\mu\nu}\,, \tag{5.32}$$

where μ, ν, ρ are SO(8, 1) indices. Therefore, in terms of the M-theory parameters, the complex three-form field strength G (5.23) can be written as

$$k_{[\rho}C^{(3)}_{\mu\nu]z} = \frac{1}{2\sqrt{\mathcal{V}_2\Omega_2}}\left(k_{[\rho}C^{(3)}_{\mu\nu]8} - \Omega k_{[\rho}C^{(3)}_{\mu\nu]11}\right), \tag{5.33}$$

and G^* is identified with $k_{[\rho}C^{(3)}_{\mu\nu]\bar{z}}$. The transverse part of the $C^{(3)}$ vertex operator (3.9) gives the vertex for transverse components of the IIB self-dual five-form field strength

$$V_{\hat{F}_5} = k_{[i}C^{(3)}_{lmn]}\left(p^i + \frac{2}{3}\mathcal{R}^{ij}k_j\right)\mathcal{R}^{lmn}e^{-ik\cdot X} \tag{5.34}$$

and those for the polarization (5.33) and its complex conjugate are given by

$$\begin{aligned} V_G &= k_{[l}C^{(3)}_{mn]z}\left(p_{\bar{z}} + \frac{2}{3}\mathcal{R}^{zj}k_j\right)\mathcal{R}^{lmn}e^{-ik\cdot X}\,, \\ V_{G^*} &= k_{[l}C^{(3)}_{mn]\bar{z}}\left(p_z + \frac{2}{3}\mathcal{R}^{\bar{z}j}k_j\right)\mathcal{R}^{lmn}e^{-ik\cdot X} \end{aligned} \tag{5.35}$$

where i, j, l, m, n are SO(7) vector indices.

We will now discuss 'protected' interaction terms that include the antisymmetric tensor C^3 field associated with the vertices (5.33) and (5.34). There are two amplitudes that only involve G and G^* with nozero U(1) weight namely G^8 and $G^5\partial^2 G^*$. It is easy to see that in the calculation of the G^8 amplitude saturating the fermionic modes picks out the $p_{\bar{z}}\mathcal{R}^{lmn}$ term of each the vertex V_G in (5.35). For the $G^5\partial^2 G^*$ amplitude a combination of four factors of the $p_{\bar{z}}\mathcal{R}^{lmn}$ term in V_G, one factor of the $\mathcal{R}^{zj}\mathcal{R}^{lmn}k_j$ term in V_G, and one from the $\mathcal{R}^{\bar{z}j}\mathcal{R}^{lmn}k_j$ term in V_{G^*} is required. These amplitudes can be calculated in the same way as the λ^{16} term and result in effective interactions of the form $f^{(4,-4)}G^8$ and $f^{(2,-2)}G^5\partial^2 G^*$, where

$$\begin{aligned} f^{(4,-4)}(\Omega,\bar{\Omega}) &= \frac{\Gamma(11/2)}{\Gamma(3/2)}\sum_{(\hat{m},\hat{n})\neq(0,0)}\Omega_2^{3/2}\frac{(\hat{m}+\hat{n}\bar{\Omega})^8}{|\hat{m}+\hat{n}\Omega|^{11}}\,, \\ f^{(2,-2)}(\Omega,\bar{\Omega}) &= \frac{\Gamma(7/2)}{\Gamma(3/2)}\sum_{(\hat{m},\hat{n})\neq(0,0)}\Omega_2^{3/2}\frac{(\hat{m}+\hat{n}\bar{\Omega})^4}{|\hat{m}+\hat{n}\Omega|^{7}}\,. \end{aligned} \tag{5.36}$$

The transformation properties of the fields and the generalized modular functions in (5.36) lead to the expected SL(2, $\mathbb{Z}$)-invariant interactions in the IIB theory.

More generally, the exact expression for all the protected IIB higher-derivative interactions that follow from the expansion of (5.27) can be deduced by considering the appropriate decompactification limit of one-loop amplitudes in compactified eleven-dimensional supergravity. The resulting interactions are proportional to

$$\int d^{10}x \sqrt{g}\, e^{-\phi^B/2}\, f^{(w,-w)}(\tau,\bar\tau)\mathcal{P}^{(2w)}\,, \tag{5.37}$$

where $\mathcal{P}^{(2w)}$ denotes an interaction between a set of fields with net U(1) charge $2w$ and

$$\begin{aligned} f^{(w,-w)}(\tau,\bar\tau) &= \frac{\Gamma(w+3/2)}{\Gamma(3/2)} \sum_{(\hat m,\hat n)\neq(0,0)} \Omega_2^{3/2}\, \frac{(\hat m+\hat n\bar\Omega)^{2w}}{|\hat m+\hat n\Omega|^{2w+3}}\,, \qquad \text{for } w\geq 0\,, \\ f^{(w,-w)}(\tau,\bar\tau) &= \frac{\Gamma(|w|+3/2)}{\Gamma(3/2)} \sum_{(\hat m,\hat n)\neq(0,0)} \Omega_2^{3/2}\, \frac{(\hat m+\hat n\Omega)^{2|w|}}{|\hat m+\hat n\Omega|^{2|w|+3}}\,, \qquad \text{for } w\leq 0\,. \end{aligned} \tag{5.38}$$

Another interesting IIB amplitude which can be obtained in this way is the amplitude involving four self-dual five-form fields $\hat F_5$. The eleven dimensional vertex (5.34) gives the amplitude which involves $(\hat F_5)_{ilmn8}$. From the self-duality of $\hat F_5$ it follows that $(\hat F_5)_{ilmn8}$ is equivalent to $(\hat F_5)_{ijlmn}$, hence the amplitude calculated in eleven dimensional supergravity on T^2 can be expressed covariantly. Indeed, the wave function $C^{(3)}_{lmn}$ in (5.34) has 35 physical degrees of freedom which is the number of physical components of the self-dual five-form $\hat F_5$. The calculation of the four $\partial\hat F_5$ amplitude is analogous to the R^4 amplitude and it is straightforward to show that the amplitude has the same modular structure (5.20). The main difference lies in the fact that the $\mathcal{R}^{ij}\mathcal{R}^{lmn}$ term in each verte (5.34) contributes to the amplitude, resulting in the following kinematic factor

$$(\partial\hat F_5)^4 = \int d^{16}\theta \Big(\bar\theta^*\Gamma^{\mu\nu\rho}\theta\bar\theta^*\Gamma^{\sigma\eta\omega}\theta\partial_\mu(\hat F_5)_{\nu\rho\sigma\eta\omega}\Big)^4 \tag{5.39}$$

Similarly terms like $(\partial\hat F_5)R^2$ and $(\partial G\partial G^*)^2$ in (5.22) can also be obtained as well as terms involving the fermionic fields. The $(\partial\hat F_5)^4$ and $(\partial\hat F_5)R^2$ vanish in the $AdS_5\times S_5$ background of IIB in which $\hat F_5$ is a constant. This is in accord with expectations based on the AdS/CFT correspondence [24, 25].

The $(\partial G\partial G^*)^2$ together with the $(\partial\hat F_5)^4$ interaction vertex in nine dimensions, has a piece which survives the decompactification limit to eleven dimensions and produces a $(\partial F_4)^4$ term in eleven dimensions. This term is a higher derivative correction of the eleven dimensional supergravity and appears at the same order as R^4. The four-point scattering amplitude of the three-form potential was calculated in [20] using the matrix theory at one-loop and identified with the classical supergravity result. It would be of interest to see how the vertex $(\partial F_4)^4$ together with other terms (such as the R^4 term) arise in matrix theory.

It would be interesting to use the eleven-dimensional vertices in situations with nonzero momentum in the eleventh dimension to describe the interactions of D-particles which should agree with results of [26] obtained by quantization of the massive ten-dimensional superparticle in static gauge, although we have not checked this. Another interesting generalization would be the formulation of covariant superspace vertex operators for the eleven-dimensional superparticle. This would be a generalization of the open-string vertex operators of [14].

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